Bounded homotopy theory and the K-theory of weighted complexes

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Abstract

Given a bounding class $\mathcal{B}$, we construct a bounded refinement $\mathcal{B}K(-)$ of Quillen's K-theory functor from rings to spaces. As defined, $\mathcal{B}K(-)$ is a functor from weighted rings to spaces, and is equipped with a comparison map $BK \to K$ induced by "forgetting control". In contrast to the situation with $\mathcal{B}$-bounded cohomology, there is a functorial splitting $\mathcal{B}K(-) \simeq K(-) \times \mathcal{B}K^{rel}(-)$ where $\mathcal{B}K^{rel}(-)$ is the homotopy fiber of the comparison map.

Contents

1 Introduction

Instead of considering all cocycles, one can restrict attention to cocycles which are bounded with respect to a *bounding class* $\mathcal{B}$. Forgetting that any condition was imposed on the cocycles gives a natural map from $\mathcal{B}$-bounded cohomology to ordinary group cohomology with coefficients—this yields a *comparison map*

$$\mathcal{B}H^\star(G;V) \to H^\star(G;V)$$

functorial in G and V. If this map is an isomorphism for all *suitable* coefficient modules V (where suitable means bornological modules over the rapid decay algebra $\mathcal{H}_{\mathcal{B},L}(G)$ as defined in [1]), we say that G is strongly $\mathcal{B}$-isocohomological (abbreviated to $\mathcal{B}$-$\mathcal{SIC}$). Properties of such comparison maps are related to geometric properties of the group; e.g., surjectivity of the comparison map is related to hyperbolicity when $\mathcal{B} = \{\text{constant functions}\}$ [3] and more general isocohomologicality is related to combings [1].

In light of the success of bounded methods in cohomology, the precedent has been set to consider $\mathcal{B}$-bounded variants of K-theory, and to introduce a K-theoretic comparison map $\mathcal{B}K^\star(G) \to K^\star(G)$. We do so in Section 3: given a bounding class $\mathcal{B}$, we construct $\mathcal{B}K(-)$, a functor from weighted rings[1] to spaces, and a comparison map

$$\mathcal{B}K(-) \to K(-)$$

which is a natural transformation on the category of weighted rings.

Although similar in appearance to the "forget control" maps of controlled K-theory (see, e.g., [5]), the bounded K-theory developed here is quite different, and in particular, is founded in what might be called *weighted algebraic topology.* In contrast to the comparison map in $\mathcal{B}$-bounded cohomology, we have (Theorem 20) a functorial splitting

$$\mathcal{B}K(-) \simeq K(-) \times \mathcal{B}K^{\text{rel}}(-)$$

[1]Although our theory applies generally to weighted rings, defined in Section 2.1.6, our primary focus in this paper will be weighted rings of the form $R[G]$, where R is a discretely normed ring and G is a group with word length.

where $\mathcal{B}K^{\mathrm{rel}}(-)$ is the homotopy fiber of the comparison map $\mathcal{B}K(-) \to K(-)$. The splitting extends to a splitting of spectra.

Given the existence of the relative theory, it is a relevant question as to whether or not it is nontrivial—in other words, are $\mathcal{B}K$-theory and K-theory actually different? There is evidence to believe that the following is true.

Conjecture 1. *Suppose a group G is of type* FL *and is not $\mathcal{B}$-$\mathcal{SIC}$. Then $[C_\star(EG)] - \chi(G)$ represents a nonzero element in $\mathcal{B}K_0^{\mathrm{rel}}(\mathbb{Z}[G])$, where EG is the homogeneous bar resolution of G.*

The relative group $\mathcal{B}K_0^{\mathrm{rel}}(\mathbb{Z}[G])$ represents a bounded version of the Wall group, and measures precisely the obstruction of a homotopically finite weighted chain complex over $\mathbb{Z}[G]$ to being homotopically finite via a $\mathcal{B}$-bounded chain homotopy. In [1], it was shown that there exists a closed 3-dimensional solvmanifold M^3 with $\pi = \pi_1 M$ for which there exists an element $t^2 \in H^2(\pi;\mathbb{C})$ not in the image of the comparison map $\mathcal{P}H^2(\pi;\mathbb{C}) \to H^2(\pi;\mathbb{C})$. Thus, as a particular case of the above conjecture, we formulate

Conjecture 2. *For $\pi = \pi_1 M^3$ as above, the class*

$$[C_\star(E\pi)] \neq 0 \in \mathcal{P}K_0^{\mathrm{rel}}(\mathbb{Z}[\pi])$$

represents an element of infinite order, where $\mathcal{P}$ is the bounding class of polynomial functions.

The theory presented here may be thought of as the "linearized" version of Waldhausen K-theory for weighted spaces, a topic we hope to address more completely in some future work. It is clear that much more needs to be said about even the groups $\mathcal{B}K_0(\mathbb{Z})$, which are at this point completely unknown even for the polynomial bounding class. This paper should be seen as an introduction to the theory.

2 Bounded homotopies of weighted chain complexes

2.1 Weighted modules and bounding classes

2.1.1 Bounding classes

We begin by recalling the definition of a bounding class [1, 2]. Let $\mathcal{S}$ denote the set of non-decreasing functions $\mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$. A collection of functions $\mathcal{B} \subset \mathcal{S}$ is *weakly closed* under the operation $\varphi : \mathcal{S}^n \to \mathcal{S}$ if, for each $(f_1, \ldots, f_n) \in \mathcal{B}^n$, there is an $f \in \mathcal{B}$ with $\varphi(f_1, \ldots, f_n) < f$. A *bounding class* is a subset $\mathcal{B} \subset \mathcal{S}$ such that

($\mathcal{B}$C1) $\mathcal{B}$ contains the constant function 1,

($\mathcal{B}$C2) $\mathcal{B}$ is weakly closed under positive rational linear combinations, and

($\mathcal{B}$C3) $\mathcal{B}$ is weakly closed under the operation $(f, g) \mapsto f \circ g$ for $f \in \mathcal{B}$ and $g \in \mathcal{L}$.

Here, $\mathcal{L}$ denotes the linear bounding class $\{f(x) = ax + b \mid a, b \in \mathbb{Q}^{\geq 0}\}$. Other examples include the polynomial bounding class $\mathcal{P}$, the bounding class $\mathcal{E}$ of simple exponential functions, and the bounding class $\tilde{\mathcal{E}}$ of iterated exponential functions. It is easy to see that any bounding class $\mathcal{B}$ can be closed under the operation of composition to form a bounding class $\mathcal{B}'$ containing $\mathcal{B}$.

A bounding class $\mathcal{B}$ is *composable* if $\mathcal{B}$ is weakly closed under the operation $(f, g) \mapsto f \circ g$ for $f, g \in \mathcal{B}$. The polynomial bounding class $\mathcal{P}$ is composable; the exponential bounding class $\mathcal{E}$ is not. Note, however, that any bounding class admits a closure under the operation of composition, and thus for any $\mathcal{B}$ there is (up to suitable equivalence) a smallest composable bounding class $\mathcal{B}'$ with $\mathcal{B} \subseteq \mathcal{B}'$.

We will write $\mathcal{B}' \preceq \mathcal{B}$ if, for every $f' \in \mathcal{B}'$, there is an $f \in \mathcal{B}$ for which $f(x) \geq f'(x)$ for all large x. Also, we write $\mathcal{B}' \prec B$ if $\mathcal{B}' \preceq \mathcal{B}$ and $\mathcal{B} \not\preceq \mathcal{B}'$.

2.1.2 Weights

A *weighted set* (X, w_X) is simply a set X with a function $w_X : X \to \mathbb{R}^{\geq 0}$. The weights are part of the data of a weighted set, but whether a morphism of weighted sets $m : (X, w_X) \to (Y, w_Y)$ is "bounded" depends on the choice of a bounding class $\mathcal{B}$; a $\mathcal{B}$-bounded set map $m : (X, w_X) \to (Y, w_Y)$ is a map for which there exists $f \in \mathcal{B}$ so that

$$w_Y(m(x)) \leq f(w_X(x))$$

for all $x \in X$.

Note that when X is finite, a morphism $m : (X, w_X) \to (Y, w_Y)$ is $\mathcal{B}$-bounded for any choice of bounding class $\mathcal{B}$ and weight function w_X on X. The distinction between "bounded" and "unbounded" only arises when the domain is an infinite set.

Weighted sets can be considered in an equivariant context. For a group G generated by S where $S = S^{-1}$, there is a natural notion of weight: a function $L : G \to \mathbb{R}^{\geq 0}$ is a *length function* if $L(gh) < L(g) + L(h)$ and $L(g) = L(g^{-1})$ for $g, h \in G$. A length function is a *word length function* if $L(1) = 1$ and there is a function $\varphi : S \to \mathbb{R}^{\geq 0}$ so that

$$L(g) = \min \left\{ \sum_{i=1}^{n} \varphi(x_i) \,\middle|\, x_i \in S,\, x_1 x_2 \cdots x_n = g \right\}.$$

Given a discrete group G with length function L, a *weighted G-set* is a weighted set (S, w_S) with a G-action on S, satisfying

$$w_S(gs) \leq C\,(L(g) + w_S(s))$$

for all $g \in G$ and $s \in S$. Analogous to the nonequivariant case, when given a bounding class $\mathcal{B}$, we may consider the $\mathcal{B}$-bounded maps of weighted G-sets.

2.1.3 Free weighted modules

We consider modules for which the elements are weighted; just as with weighted sets, for each bounding class $\mathcal{B}$, we may consider $\mathcal{B}$-bounded morphisms.

Definition 1. Let R be a normed ring (in applications, R will often be $\mathbb{Z}$). Given a weighted set (S, w_S), the free R-module $R[S]$ receives a seminorm for every $f \in \mathcal{B}$, via

$$\left\| \sum_{s \in S} \alpha_s s \right\|_f = \sum_{s \in S} |\alpha_s| \, f(w_S(s)).$$

With this setup, we call $R[S]$ a *weighted R-module.* If (S, w_S) is a weighted G-set, then $R[S]$ has the additional structure of a *weighted $R[G]$-module*; again, for any bounding class $\mathcal{B}$, the $R[G]$-module $R[S]$ can be equipped with a collection of seminorms indexed by $\mathcal{B}$.

One particular example will be important in applications: a *free weighted $R[G]$-module* is a module of the form $R[G][X] = R[G \times X]$, where X is a weighted set (X, w_X), and $G \times X$ is the weighted G-set with weight

$$w_{G \times X}(g, x) = L(g) + w_X(x).$$

If $R[G][X_1]$ and $R[G][X_2]$ are two such free weighted $R[G]$-modules, then their direct sum $R[G][X_1] \oplus R[G][X_2]$ is again a free weighted $R[G]$-module via the identification

$$R[G][X_1] \oplus R[G][X_2] \cong R[G][X_1 \sqcup X_2]$$

where the weight function on $X_1 \sqcup X_2$ is the obvious one whose restriction to X_i is w_{X_i}.

Given two free weighted $R[G]$-modules, a natural next step is to consider bounded maps between them—but bounded in what sense? A map of free $R[G]$-modules $\varphi : R[G][X] \to R[G][Y]$ is *$\mathcal{B}$-bounded (in the sense of Dehn functions)* if there exists $f \in \mathcal{B}$ so that for all $a \in R[G][X]$

$$\|\varphi(a)\|_{\mathrm{id}} \leq f\left(\|a\|_{\mathrm{id}}\right)$$

where $\|-\|_{\mathrm{id}}$ means the weighted ℓ_1-norm

$$\left\| \sum_i r_i s_i \right\|_{\mathrm{id}} = \sum_i |r_i| \, w_S(s_i).$$

Alternatively, we say that $\varphi : R[G][X] \to R[G][Y]$ is *$\mathcal{B}$-bounded (in the sense of functional analysis)* if, for every $f \in \mathcal{B}$, there exists an $f' \in \mathcal{B}$, so that for all $x \in R[G][X]$ the inequality $\|\varphi(x)\|_f < \|x\|_{f'}$ holds.

This second notion (boundedness in the functional analytic sense) is in general stronger than the first, but there are situations in which these two notions agree. For example, a $\mathcal{B}$-bounded map of sets $m : (X, w_X) \to (Y, w_Y)$ induces a map $R[G][X] \to R[G][Y]$ which is $\mathcal{B}$-bounded in either of the two senses. Under mild hypotheses on R and on the bounding class $\mathcal{B}$, the same is true for not necessarily based maps.

Lemma 2. *Consider two weighted sets (X, w_X) and (Y, w_Y), and suppose R is a normed ring, with the norm $\|-\| : R \to (\epsilon, \infty)$ where $\epsilon > 0$, and $\varphi : R[G][X] \to R[G][Y]$ is an $R[G]$-module map which is $\mathcal{B}$-bounded in the sense of Dehn functions. If $\mathcal{B} \succeq \mathcal{L}$, then φ is $\mathcal{B}$-bounded in the sense of functional analysis.*

Proof. By assumption, there exists $f \in \mathcal{B}$ so that $\|\varphi(a)\|_{\mathrm{id}} \leq f\left(\|a\|_{\mathrm{id}}\right)$ for all $a \in R[G][X]$. One then verifies the following two claims.

Claim 1. $\|\varphi(a)\|_h \leq (h \circ f)(\|a\|_{\mathrm{id}})$.

Evaluating φ on basis elements shows $\|\varphi(gx)\|_h \leq (h \circ f)(\|gx\|_{\mathrm{id}}) = \|gx\|_{h \circ f}$.

Claim 2. For a general element $a = \sum \lambda_{g,x} gx$ one has a sequence of inequalities

$$\begin{aligned} \|\varphi(a)\|_h &\leq \sum_{g,x} |\lambda_{gx}| \cdot \|\varphi(gx)\|_h \\ &\leq \sum_{g,x} |\lambda_{gx}| \cdot \|gx\|_{h\circ f} \\ &\leq \|a\|_{h \circ f}\,. \end{aligned}$$

The arguments for these two claims are as given in Lemma 1 of [1]. □

In light of Lemma 2, we will assume that $\mathcal{B} \succeq \mathcal{L}$ for the remainder of this paper. When we speak of $\mathcal{B}$-boundedness without any qualification, we mean $\mathcal{B}$-bounded in the functional analytic sense; this is a more natural notion from the bornological perspective.

For maps between not necessarily free weighted $R[G]$-modules, the relationship between the two notions of boundedness is less clear, but we do have $\mathcal{B}$-boundedness for an important class of morphisms.

Proposition 3. *Let S be a weighted G-set (on which the G-action is not necessarily free), and R a normed ring, with the module $M = R[S]$ having a family of seminorms coming from a bounding class $\mathcal{B}$, as in Definition 1. Then left multiplication by any element of $R[G]$ is $\mathcal{B}$-bounded.*

Proof (compare to Proposition 1 in [1]): Suppose $a = \sum_{g \in G} a_g\, g \in R[G]$ and $b = \sum_{s \in S} b_s\, s \in R[S]$. Given $f \in \mathcal{B}$, choose $f_2 \in \mathcal{B}$ so that $f_2(x) \geq f(2x)$ and choose $F \in \mathcal{B}$ so that $F(x) \geq \max\{x, f_2(x)\}$. Then

$$
\begin{aligned}
\|ab\|_f &= \left\| \left(\sum_{g \in G} a_g\, g \right) \left(\sum_{s \in S} b_s\, s \right) \right\|_f \\
&= \sum_{s \in S} \left| \sum_{gs' = s} a_g\, b_{s'} \right| f(w(s)) \\
&\leq \sum_{s \in S} \sum_{gs' = s} |a_g\, b_{s'}|\, f\left(L(g) + w(s')\right) \\
&\leq \sum_{s \in S} \sum_{\substack{gs' = s \\ L(g) \leq w(s')}} |a_g\, b_{s'}|\, f(2w(s')) + \sum_{s \in S} \sum_{\substack{gs' = s \\ L(g) \geq w(s')}} |a_g\, b_{s'}|\, f(2L(g)) \\
&\leq \left\| \sum_{g \in G} a_g g \right\|_1 \left\| \sum_{s \in s} b_s s \right\|_{f_2} + \left\| \sum_{g \in G} a_g g \right\|_{f_2} \left\| \sum_{s \in S} b_s s \right\|_1 \\
&\leq 2 \left\| \sum a_g g \right\|_F \left\| \sum b_s s \right\|_F = 2 \|a\|_F \cdot \|b\|_F \,.
\end{aligned}
$$

□

Proposition 3 is true even when X is an infinite set; in the case of maps between finitely generated weighted modules, much more is true.

Proposition 4. *Let $\mathcal{B}$ be a bounding class, G a group with word length, and X resp. Y finite weighted sets. Then every $R[G]$-module map $h : R[G][X] \to R[G][Y]$ is $\mathcal{B}$-bounded.*

Proof. The sets X and Y are finite; enumerate these sets, $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. We regard h as an n-by-m matrix (h_{ij}) with entries in $R[G]$.
Given $\alpha = \sum_{g \in G} \sum_{i=1}^{n} a_{g,x_i}\, g\, x_i \in R[G][X]$,

$$
h(\alpha) = \sum_{g \in G} \sum_{i=1}^{n} a_{g,x_i}\, g\, h(x_i) = \sum_{g \in G} \sum_{i=1}^{n} \sum_{j=1}^{m} a_{g,x_i}\, g\, h_{ij} y_j.
$$

For $f \in \mathcal{B}$, choose $f_2 \in \mathcal{B}$ and $f_4 \in \mathcal{B}$ so that $f_2(x) \geq f(2x)$ and $f_4(x) \geq f(4x)$. Then,

$$\begin{aligned}
\|h(\alpha)\|_f &= \left\| \sum_{g\in G} \sum_{i=1}^{n} \sum_{j=1}^{m} a_{g,x_i}\, g\, h_{ij} y_j \right\|_f \\
&\leq \sum_{j=1}^{m} \left\| \sum_{g\in G} \sum_{i=1}^{n} a_{g,x_i}\, g\, h_{ij} y_j \right\|_f \\
&\leq \sum_{j=1}^{m} 2 \left\| \sum_{g\in G} \sum_{i=1}^{n} a_{g,x_i}\, g\, h_{ij} \right\|_{f_2} \|y_j\|_{f_2} \\
&\leq C_f \sum_{j=1}^{m} \left\| \sum_{g\in G} \sum_{i=1}^{n} a_{g,x_i}\, g\, h_{ij} \right\|_{f_2} \\
&\leq C_f \sum_{j=1}^{m} 2 \sum_{g\in G} \sum_{i=1}^{n} \|a_{g,x_i}\, g\|_{f_4} \|h_{ij}\|_{f_4} \\
&\leq 2C_f \sum_{g\in G} \sum_{i=1}^{n} \|a_{g,x_i}\, g\|_{f_4} H_{f_4} \\
&= 2\, C_f\, H_{f_4} \sum_{g\in G} \sum_{i=1}^{n} \|a_{g,x_i}\, g\|_{f_4} \\
&\leq 2\, C_f\, H_{f_4}\, C \sum_{g\in G} \sum_{i=1}^{n} \|a_{g,x_i}\, g\, x_i\|_{f_4} \\
&= 2\, C_f\, H_{f_4}\, C\, \|\alpha\|_{f_4},
\end{aligned}$$

where $C_f = \max_j \|y_j\|_{f_2}$, $H_{f_4} = \max_{i,j} \|h_{ij}\|_{f_4}$, and $C = \max_i 1/w_X(x_i)$. □

More generally, Proposition 4 holds for infinite sets X and Y and any $R[G]$-module map represented by a matrix with finitely many non-zero entries.

2.1.4 Projective weighted modules and admissible maps

Definition 5. A *weighted projective* $R[G]$*-module* is a pair (M, p), where M is a weighted free $R[G]$-module, and $p : M \to M$ is an $\mathcal{L}$-bounded projection map (meaning $p^2 = p$) which admits an $\mathcal{L}$-bounded section (recall that $\mathcal{L}$ denotes the bounding class of non-decreasing linear functions).

By Proposition 4, if M is finitely generated as an $R[G]$-module, then any projection $p : M \to M$ is $\mathcal{L}$-bounded and admits an $\mathcal{L}$-bounded section.

In general, given a free weighted $R[G]$-module $R[G][X]$ and a submodule $M \subset R[G][X]$, the quotient $R[G][X]/M$ inherits an obvious weighting—called the *induced weighting*—from the weighting on $R[G][X]$, by defining the weight of an element to be the infimum among representatives. This convention for weighting quotients of free weighted modules extends to direct sums: if M_i is a submodule of $R[G][X_i]$ for $i = 1, 2$, then the direct sum

$$R[G_1]/M_1 \oplus R[G][X_2]/M_2$$

inherits a weighting via identification with the quotient of $R[G][X_1] \oplus R[G][X_2]$ by the submodule $M_1 \oplus M_2$.

Given two weighted projective $R[G]$-modules, say (M, p) and (N, q), a map $f : (M, p) \to (N, q)$ consists of a map $f : M \to N$ which intertwines with the projections p and q, i.e., a map f so that

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{f} & N \end{array}$$

commutes.

By Proposition 4, any morphism between finitely generated weighted projective $R[G]$-modules is bounded. This need not be the case for non-finitely-generated weighted projective $R[G]$-modules.

Definition 6. An epimorphism $M \twoheadrightarrow N$ is *admissible* if it admits a linearly bounded section; a monomorphism $f : M \hookrightarrow N$ is admissible if the projection $N \to (\text{cofiber } f)$ is admissible. Here the cofiber of f has the induced weighting.

Admissibility guarantees that $\mathcal{B}\operatorname{Hom}_{\mathbb{Z}[G]}(-, V)$ sends a short exact sequence with admissible maps to a short exact sequence. With this restricted class of monomorphisms and epimorphisms, the larger category of not necessarily finitely generated weighted modules over the weighted ring $R[G]$ is an exact category.

2.1.5 Categories of Modules

The various modules we study can be packaged together into categories.

Definition 7. We summarize the categories we will be using.

- $\mathbf{F}(R[G])$ and $\mathbf{P}(R[G])$ denote the categories of free and projective $R[G]$-modules, respectively, with $R[G]$-module maps.
- $\mathbf{F}_{\mathrm{f}}(R[G])$ and $\mathbf{P}_{\mathrm{f}}(R[G])$ denote the categories of finitely generated free and finitely generated projective $R[G]$-modules, respectively, with $R[G]$-module maps.
- $\mathbf{F}^{\mathrm{w}}(R[G])$ and $\mathbf{P}^{\mathrm{w}}(R[G])$ denote the categories of weighted free and weighted projective $R[G]$-modules, respectively, with (not necessarily bounded) $R[G]$-module maps.

- $\mathcal{B}\mathbf{F}^{\mathrm{w}}(R[G])$ and $\mathcal{B}\mathbf{P}^{\mathrm{w}}(R[G])$ denote the categories of weighted free and weighted projective $R[G]$-modules, respectively, with $\mathcal{B}$-bounded $R[G]$-module maps. For this to form a category, the morphisms need to be composable, which requires that the bounding class $\mathcal{B}$ be composable.

Unlike the first three cases, $\mathcal{B}\mathbf{F}^{\mathrm{w}}(R[G])$ and $\mathcal{B}\mathbf{P}^{\mathrm{w}}(R[G])$ are almost never abelian categories, even if $\mathcal{B}$ is the bounding class $\mathcal{B}_{\max}$ of all non-decreasing functions. But nevertheless, in each of these categories, there is a notion of *zero morphism*, so one can construct chain complexes of objects in these categories.

2.1.6 Modules over weighted rings

The above structures can be codified by the notion of a *weighted ring*, meaning a ring $\tilde{R}$ equipped with two norms: an ℓ_1 norm and a weighted ℓ_1 norm (corresponding to the weighted ℓ_1 norm coming from the word length function on G).

A *weighted $\tilde{R}$-module M* is an $\tilde{R}$-module similarly equipped with a pair of norms $\|-\|_1$ and $\|-\|_{\mathrm{w}}$ satisfying

$$\|r \cdot m\|_1 \leq \|r\|_1 \cdot \|m\|_1$$

and also (as in the proof of Proposition 3)

$$\|r \cdot m\|_w \leq \|r\|_{\mathrm{w}} \|m\|_1 + \|r\|_1 \|m\|_{\mathrm{w}} .$$

A weighted ring $\tilde{R}$ is required to be an $\tilde{R}$-module, with respect to both left and right multiplication.

So defined, the K-theoretic constructions introduced in the following sections can be extended in a natural way to the more general class of weighted rings. However, for the purpose of this paper, we will assume henceforth that our weighted rings $\tilde{R}$ are of the form $R[G]$ for a normed ring R and a discrete group G with word length function.

2.2 Categories of complexes

Now we consider categories of chain complexes of weighted $R[G]$-modules; let $\mathbf{C}$ denote one of the aforementioned categories with zero morphisms (e.g., $\mathbf{F}(R[G])$, $\mathbf{P}(R[G])$, $\mathbf{F}_{\mathrm{f}}(R[G])$, $\mathbf{P}_{\mathrm{f}}(R[G])$, $\mathbf{F}^{\mathrm{w}}(R[G])$, $\mathbf{P}^{\mathrm{w}}(R[G])$, $\mathcal{B}\mathbf{F}^{\mathrm{w}}(R[G])$, or $\mathcal{B}\mathbf{P}^{\mathrm{w}}(R[G])$). The objects of the category $\mathbf{Ch}(\mathbf{C})$ are the *chain complexes* of objects in the category $\mathbf{C}$; the differentials in the chain complex are morphisms in $\mathbf{C}$, and the morphisms between objects of $\mathbf{Ch}(\mathbf{C})$ are the *chain maps*.

There are many variants of this construction: one may impose finiteness conditions (e.g., one can demand that the chain complexes be finite, or merely chain homotopy equivalent to a finite complex), and, when the objects are weighted, one may demand that certain aspects of the chain complexes be $\mathcal{B}$-bounded (e.g., that the differentials, the chain maps, or the chain homotopies be bounded). Notation for describing combinations of these conditions is summarized in the following definition.

Definition 8. The following are subcategories of $\mathbf{Ch}(\mathbf{C})$.

- $\mathbf{Ch}_{\text{fin}}(\mathbf{C})$ denotes the full subcategory of chain complexes which are *finite*; a chain complex $(A_\star, d)$ is finite if $\bigoplus_{n\in\mathbb{Z}} A_n$ is finitely generated over $R[G]$.
- $\mathbf{Ch}_{h\text{fin}}(\mathbf{C})$ denotes the full subcategory of *homotopically finite* chain complexes; a chain complex is homotopically finite if it is chain homotopy equivalent to a finite chain complex.

If $\mathbf{C}$ is a category with weighted objects (e.g., $\mathbf{F}^{\mathrm{w}}(R[G])$, $\mathbf{P}^{\mathrm{w}}(R[G])$, $\mathcal{B}\mathbf{F}^{\mathrm{w}}(R[G])$, or $\mathcal{B}\mathbf{P}^{\mathrm{w}}(R[G])$) and $\mathcal{B}$ is a bounding class, then there are "bounded" subcategories of $\mathbf{Ch}(\mathbf{C})$ worth considering.

- The categories $\mathcal{B}\mathbf{Ch}(\mathbf{C})$, $\mathcal{B}\mathbf{Ch}_{\text{fin}}(\mathbf{C})$, and $\mathcal{B}\mathbf{Ch}_{h\text{fin}}(\mathbf{C})$ have the same objects as $\mathbf{Ch}(\mathbf{C})$, $\mathbf{Ch}_{\text{fin}}(\mathbf{C})$, and $\mathbf{Ch}_{h\text{fin}}(\mathbf{C})$, respectively, but the morphisms in categories prefixed by $\mathcal{B}$ are, degreewise, $\mathcal{B}$-bounded.
- $\mathcal{B}\mathbf{Ch}_{\mathcal{B}h\text{fin}}(\mathbf{C})$ is a full subcategory of $\mathcal{B}\mathbf{Ch}_{h\text{fin}}(\mathbf{C})$; the objects of $\mathcal{B}\mathbf{Ch}_{\mathcal{B}h\text{fin}}(\mathbf{C})$ are chain homotopy equivalent to a finite complex via a $\mathcal{B}$-bounded chain homotopy.

Observation 9. To understand how this notation is being used, one can consider the difference between $\mathcal{B}\mathbf{Ch}(\mathbf{F}^{\mathrm{w}}(R[G]))$ and $\mathbf{Ch}(\mathcal{B}\mathbf{F}^{\mathrm{w}}(R[G]))$. In the former category, the modules are not necessarily finitely generated, the chain complexes may have differentials which are not $\mathcal{B}$-bounded, but the chain maps are $\mathcal{B}$-bounded. In the latter category the differentials, being maps in $\mathcal{B}\mathbf{F}^{\mathrm{w}}(R[G])$, are $\mathcal{B}$-bounded, but the chain maps need not be $\mathcal{B}$-bounded.

There are some obvious relationships between the above categories.

Observation 10. By Proposition 4, the forgetful functor

$$\mathcal{B}\mathbf{Ch}_{\text{fin}}(\mathcal{B}\mathbf{P}^{\mathrm{w}}(R[G])) \to \mathbf{Ch}_{\text{fin}}(\mathbf{P}^{\mathrm{w}}(R[G]))$$

is an isomorphism of categories, as every chain map in $\mathbf{Ch}_{\text{fin}}(\mathbf{P}^{\mathrm{w}}(R[G]))$ is bounded (notice the subscript "fin" forces the chain complexes to be, degreewise, finitely generated).

If X is a finite set, any two different weight functions w_1 and w_2 on X produce weighted $R[G]$-modules $R[G][(X, w_1)]$ and $R[G][(X, w_2)]$ which are, via the identity on X, canonically $\mathcal{B}$-boundedly isomorphic. Consequently, the forgetful functor

$$\mathbf{Ch}_{\text{fin}}(\mathbf{P}^{\mathrm{w}}(R[G])) \to \mathbf{Ch}_{\text{fin}}(\mathbf{P}(R[G]))$$

is an equivalence of categories.

Without the finiteness condition, $\mathbf{Ch}(\mathbf{P}^{\mathrm{w}}(R[G]))$ and $\mathbf{Ch}(\mathbf{P}(R[G]))$ are not equivalent categories.

These above observations apply for free modules in place of projective modules.

In the subcategories of $\mathcal{B}\mathbf{Ch}(\mathbf{C})$, two objects can be chain homotopy equivalent in two different ways: there is the usual ("coarse") notion of chain equivalence, and the finer relation of $\mathcal{B}$-bounded chain equivalence. Waldhausen's setup of a category with cofibrations and weak equivalences axiomatizes the comparison of equivalences on a category; we review his setup now.

3 Waldhausen K-theory of $\mathcal{B}$-bounded chain complexes

3.1 Recap of Waldhausen K-theory

3.1.1 Waldhausen categories

A *Waldhausen category* (see [6]) involves two distinguished classes of morphisms: cofibrations and weak equivalences. After the definition, we explain why these distinguished classes exist in the categories discussed in Section 2.2.

Definition 11. A *category with cofibrations* means a category $\mathbf{C}$, equipped with a zero object $*$ (both initial and terminal), together with a subcategory $\operatorname{co}\mathbf{C}$, the morphisms of which are called *cofibrations*, and are denoted by hooked arrows $\hookrightarrow$. The subcategory $\operatorname{co}\mathbf{C}$ is *wide*, meaning that every object of $\mathbf{C}$ is an object of $\operatorname{co}\mathbf{C}$, but not every morphism is a cofibration.

The subcategory of cofibrations satisfies the following three properties.

(Cof 1) Every isomorphism in $\mathbf{C}$ is in $\operatorname{co}\mathbf{C}$; in short, $\operatorname{co}\mathbf{C}$ is replete.

(Cof 2) For every object X in $\mathbf{C}$, the map $* \to X$ is in $\operatorname{co}\mathbf{C}$.

(Cof 3) Cofibrations are preserved under co-base change, meaning that for any cofibration $i : X \hookrightarrow Y$ and any morphism $f : X \to Z$ in $\mathbf{C}$, there is a pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ Y & \longrightarrow & W \end{array}$$

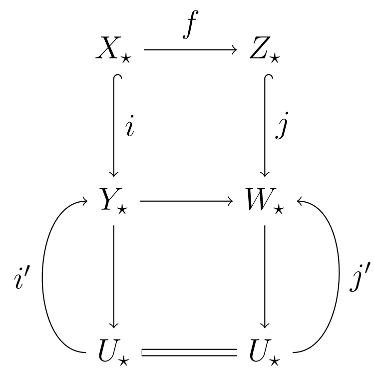
and the map $j : Z \hookrightarrow W$ is a cofibration.

Let $\mathbf{C}$ be one of the categories of modules listed above in Definition 7. In $\mathbf{Ch}(\mathbf{C})$, in the unweighted setting, a cofibration is a degreewise monomorphism of chain complexes which is degreewise split.

If $\mathbf{C}$ is a category with weighted objects and $\mathcal{B}$-bounded maps, a chain map $f : C_\star \to D_\star$ of weighted complexes in $\mathbf{Ch}(\mathbf{C})$ is a cofibration which is degreewise an admissible monomorphism, meaning that there is a choice of cofiber $E_\star = D_\star / C_\star$ so that for all $n \in \mathbb{Z}$ the map $D_n \to E_n$ is an admissible epimorphism. This yields a splitting degreewise, but not necessarily a splitting on the level of chain complexes.

Lemma 12. *Let $\mathbf{C}$ be one of the categories of chain complexes listed in Definition 8; then axioms (Cof 1), (Cof 2), and (Cof 3) of the preceding definition hold for the subcategory $\operatorname{co}\mathbf{C}$.*

Proof. For the classical case in which the chain maps are not weighted, the proof is standard. When the objects are weighted and the morphisms are $\mathcal{B}$-bounded, properties (Cof 1) and (Cof 2) are again clear; the remaining issue is (Cof 3). The fact that it is a cofibration diagram comes for free; moreover, if f and i are bounded, then j is bounded where W has the induced weighting $(Z \oplus Y)/\sim$. However, we need to know that j is a cofibration, i.e., an admissible monomorphism. Consider the diagram of weighted chain complexes

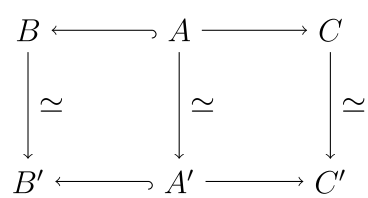


Since i is a cofibration, it is an admissible monomorphism, so there is a degree-preserving section i' of graded modules from its cofiber U. The top square is a pushout, so the cofiber of i is the same as the cofiber of j, and we get the required section j' of graded modules by following the diagram; thus, j is an *admissible* monomorphism. □

Definition 13. Given a category $\mathbf{C}$ with a subcategory $\mathrm{co}\,\mathbf{C}$ of cofibrations, a *category of weak equivalences* for $\mathbf{C}$ is a subcategory $\mathrm{w}\mathbf{C}$ which satisfies two properties.

(Weq 1) Every isomorphism in $\mathbf{C}$ is in $\mathrm{w}\mathbf{C}$.

(Weq 2) Weak equivalences can be glued together, meaning that if



where the arrows decorated with $\simeq$ are in $\mathrm{w}\mathbf{C}$, then the induced map between pushouts $B \cup_A C \to B' \cup_A C'$ is also in $\mathrm{w}\mathbf{C}$.

Again, the subcategory $\mathrm{w}\mathbf{C}$ is wide, meaning that every object in $\mathbf{C}$ is in the subcategory of weak equivalences.

Let $\mathbf{C}$ be one of the categories of modules listed in Definition 7. Consider the subcategory $h\,\mathbf{Ch}(\mathbf{C})$ which has the same objects as $\mathbf{Ch}(\mathbf{C})$ but whose morphisms are chain homotopy equivalences; doing so endows $\mathbf{Ch}(\mathbf{C})$ with the structure of a category with weak equivalences. This is the classical case. In the presence of weighted objects and a bounding class

$\mathcal{B}$, there is a finer notion of $\mathcal{B}$-bounded chain homotopy equivalence, denoted $\mathcal{B}h$. To say a chain map $F : C_\star \to D_\star$ is a $\mathcal{B}$-bounded chain homotopy equivalence means that there is a $\mathcal{B}$-bounded chain homotopy inverse $G : D_\star \to C_\star$ so that $F \circ G$ and $G \circ F$ are $\mathcal{B}$-boundedly homotopic to the identity, i.e., the chain homotopy is a $\mathcal{B}$-bounded map.

To summarize; given a category $\mathbf{C}$ of weighted objects, there are three increasingly restrictive ways of introducing weak equivalences:

- $h\,\mathbf{Ch}(\mathbf{C})$, in which weak equivalences are chain homotopy equivalences, and the weights are simply ignored;
- $h\,\mathcal{B}\mathbf{Ch}(\mathbf{C})$, in which weak equivalences are again possibly unbounded chain homotopy equivalences, but the chain maps are $\mathcal{B}$-bounded;
- $\mathcal{B}h\,\mathcal{B}\mathbf{Ch}(\mathbf{C})$, in which the weak equivalences are $\mathcal{B}$-bounded chain maps, for which the homotopies to the identity are also $\mathcal{B}$-bounded.

Lemma 14. *Axioms (Weq 1) and (Weq 2) are satisfied in the aforementioned categories.*

Proof. In the unweighted cases $h\,\mathbf{Ch}(\mathbf{C})$ and $h\,\mathcal{B}\mathbf{Ch}(\mathbf{C})$, this result is classical. In the weighted case, (Weq 1) is satisfied because a $\mathcal{B}$-bounded isomorphism is, after forgetting the weights, an isomorphism.

To verify axiom (Weq 2), we note that the weighting as defined on $(B \oplus C/\sim)$ and $(B' \oplus C'/\sim)$ produces a suitably bounded map of pushouts $(B \oplus C/\sim) \to (B' \oplus C'/\sim)$.

To see that this map is a $\mathcal{B}$-bounded chain homotopy equivalence, it suffices to verify the following technical fact:

Claim 15. *Given an admissible short exact sequence of weighted chain complexes*

$$A_\star \hookrightarrow B_\star \twoheadrightarrow C_\star$$

with $A_\star \simeq *$ via a $\mathcal{B}$-bounded chain homotopy, the admissible epimorphism $B_\star \twoheadrightarrow C_\star$ is a $\mathcal{B}$-bounded chain homotopy equivalence.

This can be verified directly using exactly the same type of argument as one uses in the unbounded case—the argument is left to the reader. □

3.1.2 K-theory of a Waldhausen category

We recall Waldhausen's $\mathrm{S}_\bullet$ construction. The poset of integers $[n] = \{0, 1, \ldots, n\}$ can be regarded as a category; the category $\mathrm{Ar}[n]$ is the category of arrows in $[n]$. Given a category $\mathbf{C}$ with cofibrations $\mathrm{co}\,\mathbf{C}$, define $\mathrm{S}_n\mathbf{C}$ to be the category of functors $A : \mathrm{Ar}[n] \to \mathbf{C}$, with two properties.

(S1) $A(j \to j) = *$

(S2) For a pair of composable arrows $i \to j$ and $j \to k$ in $\mathrm{Ar}[n]$, the map

$$\iota_{jk} : A(i \to j) \longrightarrow A(i \to k)$$

is a cofibration, and fits into a commuting diagram

$$\begin{array}{ccc} A(i \to j) & \xrightarrow{\ \iota_{jk}\ } & A(i \to k) \\ \big\downarrow & & \big\downarrow \\ A(j \to j) = * & \longrightarrow & A(j \to k) \end{array}$$

The morphisms in the category $\mathrm{S}_n\mathbf{C}$ are the natural transformations between such functors. By collecting together (for varying n) the categories $\mathrm{S}_n\mathbf{C}$, we form a simplicial category $\mathrm{S}_\bullet\mathbf{C}$.

Canonically, $\mathrm{S}_n\mathbf{C}$ can be given the structure of a Waldhausen category. In particular, given a subcategory of weak equivalences $\mathrm{w}\mathbf{C}$, the category $\mathrm{S}_n\mathbf{C}$ also has a subcategory of weak equivalences $\mathrm{wS}_n\mathbf{C}$; a natural transformation $A \to A'$ is a weak equivalence if it is a weak equivalence objectwise. In this way, one may form the basepointed simplicial space

$$\mathrm{wS}_\bullet\mathbf{C} := \{[n] \mapsto |\mathrm{wS}_n\mathbf{C}|\}_{n \geq 0}$$

The *Waldhausen K-theory space* $K(\mathbf{C})$ of $\mathbf{C}$ is defined to be $\Omega\,|\mathrm{wS}_\bullet\mathbf{C}|$, which admits a canonical delooping; we denote the associated spectrum by $\mathbb{K}(\mathbf{C})$. The homotopy groups of $\Omega\,|\mathrm{wS}_\bullet\mathbf{C}|$ are the higher K-groups of the Waldhausen category $\mathbf{C}$.

3.1.3 Approximation Theorem

Among the tools developed by Waldhausen in [6] to study his eponymous categories is his Approximation Theorem; stating this powerful theorem, however, requires introducing some additional properties that an arbitrary Waldhausen category may or may not satisfy: these are the Saturation Axiom, and the Cylinder Axiom.

Saturation Axiom. If f, g are composable maps in $\mathbf{C}$, and two of the three maps f, g, and $g \circ f$ are in $\mathrm{w}\mathbf{C}$, then the third is as well.

Lemma 16. *The categories with weak equivalences, $h\,\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$, $h\,\mathbf{Ch}_{h\mathrm{fin}}(\mathbf{C})$, $\mathcal{B}h\mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$, $\mathcal{B}h\mathcal{B}\mathbf{Ch}_{\mathcal{B}h\mathrm{fin}}(\mathbf{C})$, satisfy the Saturation Axiom.*

Proof. If f and g are weak equivalences, then clearly so is $g \circ f$ in any of these categories.

Suppose $f : A_\star \to B_\star$ and $g : B_\star \to C_\star$ are composable maps, and that f and $g \circ f$ are weak equivalences.

By $\widetilde{\mathrm{Cone}}(g \circ f)_\star$ and $\widetilde{\mathrm{Cyl}}(g \circ f)_\star$ we mean the pushouts

$$\begin{array}{ccc} B_\star & \hookrightarrow & \mathrm{Cone}(f)_\star \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(g)_\star & \twoheadrightarrow & \widetilde{\mathrm{Cone}}(g\circ f)_\star \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B_\star & \hookrightarrow & \mathrm{Cyl}(f)_\star \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(g)_\star & \twoheadrightarrow & \widetilde{\mathrm{Cyl}}(g\circ f)_\star \end{array}$$

respectively. One then has the following diagram:

$$\begin{array}{ccccc} A_\star & = & A_\star & & \\ \downarrow & & \downarrow & & \\ \mathrm{Cyl}(f)_\star & \hookrightarrow & \widetilde{\mathrm{Cyl}}(g\circ f)_\star & & \\ \downarrow & & \downarrow & & \\ \mathrm{Cone}(f)_\star & \hookrightarrow & \widetilde{\mathrm{Cone}}(g\circ f)_\star & \twoheadrightarrow & \mathrm{Cone}(g)_\star \\ \wr\| & & \wr\| & & \\ * & & * & & \end{array}$$

where the bottom row represents an admissible short exact sequence of complexes. The fact that f is a weak equivalence implies $\mathrm{Cone}(f)_\star \simeq *$ in either the bounded or unbounded setting, and similarly the fact that $g \circ f$ is a weak equivalence implies $\widetilde{\mathrm{Cone}}(g \circ f)_\star \simeq *$ in either the bounded or unbounded setting. For unbounded complexes, this immediately implies that the cokernel of the bottom row is contractible. In the $\mathcal{B}$-bounded case, we appeal to Claim 15 above to conclude that $\mathrm{Cone}(g)_\star$ is $\mathcal{B}$-boundedly contractible, implying that g is a weak equivalence.

The final case, when g and $g \circ f$ are weak equivalences, follows in the same manner. □

Before stating the Approximation Theorem, there are two more definitions that we need.

Definition 17. Let $\mathbf{C}$ be a category with cofibrations and weak equivalences, and $\mathrm{Ar}\,\mathbf{C}$ the category of arrows in $\mathbf{C}$. A *cylinder functor* on $\mathbf{C}$ is a functor from $\mathrm{Ar}\,\mathbf{C}$ to diagrams in $\mathbf{C}$, sending $f : A \to B$ to a diagram

$$\begin{array}{ccccc} A & \xrightarrow{j_1} & \mathrm{Cyl}(f) & \xleftarrow{j_2} & B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle =} & \\ & & B & & \end{array}$$

The object $\mathrm{Cyl}(f)$ is the *cylinder* of f with j_1 and j_2 corresponding to the *front inclusion* and *back inclusion*, respectively, and p corresponding to the natural *projection* to B. The maps j_1 and j_2 are in $\mathrm{co}\,\mathbf{C}$. Moreover, the functor must satisfy the following.

(Cyl 1) The front and back inclusions assemble to an exact functor $\operatorname{Ar}\mathbf{C} \to \mathrm{F}_1\,\mathbf{C}$ sending $f : A \to B$ to $j_1 \vee j_2 : A \vee B \hookrightarrow \mathrm{Cyl}(f)$. The definition of $\mathrm{F}_1\,\mathbf{C}$ can be found in [6].

(Cyl 2) $\mathrm{Cyl}(* \to A) = A$ for every object A in $\mathbf{C}$; the two inclusions and projection map in the corresponding diagram are all the identity map on A.

Cylinder Axiom. For every $f : A \to B$ in $\mathbf{C}$, the projection $p : \mathrm{Cyl}(f) \to B$ is in $\mathrm{w}\mathbf{C}$.

Lemma 18. *The categories $\mathcal{B}h\,\mathbf{Ch}_{\mathrm{fin}}$ and $h\,\mathbf{Ch}_{\mathrm{fin}}$ satisfy the Cylinder Axiom.*

Proof. Given a chain map $f : C_\star \to D_\star$, define $\mathrm{Cyl}(f) := \mathrm{Cyl}(f)_\star$ to be the algebraic mapping cylinder; in this case, the projection p is a projection onto a summand, and therefore is bounded by virtue of the way that direct sums of weighted complexes are weighted. □

Definition 19. Let $\mathbf{C}$ and $\mathbf{D}$ be categories with cofibrations and weak equivalences. A functor $F : \mathbf{C} \to \mathbf{D}$ is an *exact functor* provided $F(*) = *$, F sends weak equivalences to weak equivalences, cofibrations to cofibrations, and F preserves the pushouts appearing in (Cof 3).

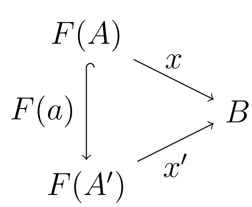
There are many examples of exact functors. For instance, the "forget control" functor $\mathbf{Ch}_{\mathcal{B}h\mathrm{fin}} \to \mathbf{Ch}_{h\mathrm{fin}}$ is exact.

The following is one of the fundamental results of Waldhausen K-theory, and a key ingredient in the proof of Theorem 20 below.

Approximation Theorem (6, Theorem 1.6.7)**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be categories with cofibrations and weak equivalences. Suppose $\mathrm{w}\mathbf{A}$ and $\mathrm{w}\mathbf{B}$ satisfy the Saturation Axiom, that $\mathbf{A}$ has a cylinder functor, and that $\mathrm{w}\mathbf{A}$ satisfies the Cylinder Axiom. Let $F : \mathbf{A} \to \mathbf{B}$ be an exact functor with the approximation properties:*

(App 1) F reflects weak equivalences, meaning that a map is a weak equivalence in $\mathbf{A}$ if and only if its image is a weak equivalence in $\mathbf{B}$.

(App 2) Given any object A in $\mathbf{A}$ and a map $x : F(A) \to B$ in $\mathbf{B}$, there exists a cofibration $a : A \hookrightarrow A'$ in $\mathbf{A}$ and a weak equivalence $x' : F(A') \to B$ in $\mathbf{B}$ for which



commutes.

Then the induced maps $|\mathrm{w}\mathbf{A}| \to |\mathrm{w}\mathbf{B}|$ and $|\mathrm{wS}_\bullet\mathbf{A}| \to |\mathrm{wS}_\bullet\mathbf{B}|$ of pointed spaces are homotopy equivalences, which extend to a map of spectra $\mathbb{K}(\mathbf{A}) \to \mathbb{K}(\mathbf{B})$.

3.2 K-theory of the $\mathcal{B}$-bounded category of complexes

Define the K-theory space of weighted complexes over $R[G]$ to be

$$\mathcal{B}K(R[G]) = \Omega\, |\mathcal{B}h\mathrm{S}_\bullet\, \mathcal{B}\mathbf{Ch}_{h\mathrm{fin}}(\mathcal{B}\mathbf{P}^{\mathrm{w}}(R[G]))|$$

with associated spectrum $\mathcal{B}\mathbb{K}(R[G])$. This is the $\mathcal{B}$-bounded analogue to the algebraic K-theory space for the ring $R[G]$:

$$K(R[G]) = \Omega\, |h\mathrm{S}_\bullet\, \mathbf{Ch}_{h\mathrm{fin}}(\mathbf{P}(R[G]))|\,.$$

There is an obvious natural transformation of infinite loop space functors

$$\mathcal{B}K(-) \to K(-)$$

induced by forgetting weights and bounds. Finally, define the relative K-theory $\mathcal{B}K^{\mathrm{rel}}(-)$ to be the homotopy fiber of $\mathcal{B}K(-) \to K(-)$.

Theorem 20. *There is a functorial splitting of infinite loop spaces*

$$\mathcal{B}K(-) \simeq K(-) \times \mathcal{B}K^{\mathrm{rel}}(-).$$

In other words, the K-theory of the category $\mathbf{Ch}_{h\mathrm{fin}}$ with respect to the weak equivalence relation of $\mathcal{B}$-bounded chain homotopy equivalence splits canonically as the product of the K-theory of $\mathbf{Ch}_{\mathrm{fin}}$ and the relative theory.

Proof. Compare the following to Proposition 2.1.1 of [6]. To conserve space, let $\mathbf{C} = \mathcal{B}\mathbf{P}^{\mathrm{w}}(R[G])$. We begin by considering the diagram[2]:

$$\begin{array}{ccc}
|\mathcal{B}h\mathrm{S}_\bullet\, \mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})| & \xrightarrow{\ \simeq\ } & |\mathcal{B}h\mathrm{S}_\bullet\, \mathcal{B}\mathbf{Ch}_{\mathcal{B}h\mathrm{fin}}(\mathbf{C})| \\
 & & \downarrow \\
\downarrow \cong & & |\mathcal{B}h\mathrm{S}_\bullet\, \mathcal{B}\mathbf{Ch}_{h\mathrm{fin}}(\mathbf{C})| \\
 & & \downarrow \\
|h\mathrm{S}_\bullet\, \mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})| & \xrightarrow{\ \simeq\ } & |h\mathrm{S}_\bullet\, \mathbf{Ch}_{h\mathrm{fin}}(\mathbf{C})|
\end{array}$$

The left-hand vertical map $|\mathcal{B}h\mathrm{S}_\bullet\, \mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})| \to |h\mathrm{S}_\bullet\, \mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})|$ is a weak homotopy equivalence; in fact, more is true: the category $\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$ is the same as the category $\mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$ by Proposition 4. Furthermore, the two choices of subcategories of weak equivalences, $\mathcal{B}h\,\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$ and $h\,\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$, are identical, so $|\mathcal{B}h\mathrm{S}_\bullet\, \mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})| \to |h\mathrm{S}_\bullet\, \mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})|$ is a homeomorphism, induced by an isomorphism of simplicial categories.

[2]This is not unlike the situation in equivariant homotopy theory, where one has various notions of weak equivalence.

The top arrow $|\mathcal{B}h\mathrm{S}_\bullet\,\mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})| \to |\mathcal{B}h\mathrm{S}_\bullet\,\mathcal{B}\mathbf{Ch}_{\mathcal{B}h\mathrm{fin}}(\mathbf{C})|$ is a weak homotopy equivalence by the Approximation Theorem; we verify properties (App 1) and (App 2). Property (App 1) is clear: $\mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$ is a full subcategory of $\mathcal{B}\mathbf{Ch}_{\mathcal{B}h\mathrm{fin}}(\mathbf{C})$; moreover, if two objects in $\mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$ are $\mathcal{B}$-boundedly chain homotopy equivalent in $\mathcal{B}\mathbf{Ch}_{\mathcal{B}h\mathrm{fin}}(\mathbf{C})$, then they were so in $\mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$. Similarly, the map $\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C}) \to \mathbf{Ch}_{h\mathrm{fin}}(\mathbf{C})$ satisfies (App 1).

The second property (App 2) is only slightly more involved. Suppose $C_\star$ is a finite weighted complex, $D_\star$ a $\mathcal{B}$-boundedly homotopically finite weighted complex, and $f : C_\star \to D_\star$ a $\mathcal{B}$-bounded chain map. Verifying (App 2) requires factoring f as

$$C_\star \overset{g}{\hookrightarrow} E_\star \overset{h}{\to} D_\star$$

with g a cofibration, and h a weak equivalence in $\mathcal{B}h\mathcal{B}\mathbf{Ch}_{\mathcal{B}h\mathrm{fin}}(\mathbf{C})$.

The chain complex $D_\star$ is $\mathcal{B}$-boundedly homotopically finite; let $j : D_\star \to D'_\star$ be a $\mathcal{B}$-bounded chain homotopy equivalence, with $D'_\star$ finite. Define $\tilde{f} = j \circ f$, and set $E_\star = \mathrm{Cyl}(\tilde{f})$. Then $E_\star$ is a finite complex, and the inclusion $g : C_\star \hookrightarrow E_\star$ is a cofibration[3] in $\mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$. The weak equivalence h is the composition of the projection $E_\star \to D'_\star$ (which is a weak equivalence) with $j^{-1} : D'_\star \to D_\star$, which is $\mathcal{B}$-bounded because its domain is finite (by Proposition 4).

The same argument, albeit without considerations of $\mathcal{B}$-boundedness, shows that

$$h\mathrm{S}_\bullet\,\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C}) \to \mathcal{B}h\mathrm{S}_\bullet\,\mathbf{Ch}_{h\mathrm{fin}}(\mathbf{C}),$$

which induces the bottom arrow after realization, satisfies (App 2).

To apply the Approximation Theorem, we also need to know the categories involved to satisfy the Saturation Axiom: this was verified in Lemma 16. Finally, the hypotheses of the Approximation Theorem require that $\mathcal{B}h\mathrm{S}_\bullet\,\mathcal{B}\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$ and $h\mathrm{S}_\bullet\,\mathbf{Ch}_{\mathrm{fin}}(\mathbf{C})$ satisfy the Cylinder Axiom: this we verified in Lemma 18.

We therefore conclude by the Approximation Theorem that the top and bottom horizontal maps are in fact weak equivalences, which in turn implies that

$$|\mathcal{B}h\mathrm{S}_\bullet\,\mathcal{B}\mathbf{Ch}_{\mathcal{B}h\mathrm{fin}}(\mathbf{C})| \longrightarrow |\mathcal{B}h\mathrm{S}_\bullet\,\mathcal{B}\mathbf{Ch}_{h\mathrm{fin}}(\mathbf{C})| \longrightarrow |h\mathrm{S}_\bullet\,\mathbf{Ch}_{h\mathrm{fin}}|$$

is a weak homotopy equivalence. Hence $|h\mathrm{S}_\bullet\,\mathbf{Ch}_{h\mathrm{fin}}(\mathbf{C})|$ splits off $|\mathcal{B}h\mathrm{S}_\bullet\,\mathcal{B}\mathbf{Ch}_{h\mathrm{fin}}(\mathbf{C})|$ up to homotopy. These maps are induced by maps of Waldhausen categories, and hence induce infinite loop space maps upon passage to K-theory. □

On the level of spectra,

$$\mathcal{B}\mathbb{K}(-) \simeq \mathbb{K}(-) \vee \mathcal{B}\mathbb{K}^{\mathrm{rel}}(-).$$

3.3 The relative Wall obstruction to $\mathcal{B}$-finiteness

Inspired by Ranicki's setup for an algebraic finiteness obstruction [4], we now consider the relationship between whether $C_\star(EG)$ vanishes in $\mathcal{B}K_0^{\mathrm{rel}}(R[G])$ and whether $C_\star(EG)$ has the $\mathcal{B}$-bounded homotopy type of a finite complex.

[3]The mapping cylinder construction together with the inclusion is the prototypical example of an admissible monomorphism in the category of $\mathcal{B}$-bounded chain complexes.

Observation 21. Let $\mathcal{B}$ be a bounding class, and let G be a group of type $\mathcal{B}\mathrm{FP}^\infty$. Then $C_\star(EG)$ is $\mathcal{B}$-boundedly homotopy equivalent to a finite complex $D_\star$, and so $[C_\star(EG)] = 0$ in $\mathcal{B}K_0^{\mathrm{rel}}(\mathbb{Z}[G])$.

By Theorem 3 in [1], a group G is of type $\mathcal{B}\mathrm{FP}^\infty$ if and only if G is $\mathcal{B}$-$\mathcal{SIC}$. The contrapositive of the observation with this result proves

Theorem 22. *Let $\mathcal{B}$ be a bounding class, and G a group of type* FP^∞. *If $[C_\star(EG)] \neq 0$ in $\mathcal{B}K_0^{\mathrm{rel}}(\mathbb{Z}[G])$, then G is not $\mathcal{B}$-$\mathcal{SIC}$.*

A concrete example from [1] may also be relevant here. Specifically, there exists a solvable group G (given as a split extension $\mathbb{Z}^2 \to G \to \mathbb{Z}$) with BG homotopy equivalent to a closed oriented 3-manifold M_G. Therefore, the manifold M_G is a finite model for BG, and via the spectral sequence constructed in [1], the group G is not $\mathcal{B}$-$\mathcal{SIC}$ for any bounding class $\mathcal{B} \prec \mathcal{E}$.

Conjecture 3. *$[C_\star(EG)]$ represents an element of infinite order in $\mathcal{P}K_0^{\mathrm{rel}}(\mathbb{Z}[G])$.*

4 An Assembly Map

We construct an assembly map

$$BG_+ \wedge \mathcal{B}\mathbb{K}(\mathbb{Z}) \to \mathcal{B}\mathbb{K}(\mathbb{Z}[G]).$$

by recognizing $\Omega^\infty\Sigma^\infty BG_+$ as the K-theory of a Waldhausen category Monomial(G), and applying Section 1.5 of [6] to promote a pairing of Waldhausen categories into a product on the level of the associated spectra.

4.1 Monomial category

Recall that a monomial matrix is a square matrix which, when conjugated by a permutation matrix, is diagonal. Define $W_n(G)$ to be the group of $n \times n$ monomial matrices with entries in $\pm G$; or to be more precise, let Σ_n denote the symmetric group on n letters. These permutations act on $n \times n$ matrices, and by interpreting $\pm G^n$ as the $n \times n$ diagonal matrices, the group Σ_n acts on $\pm G^n$ giving rise to the semidirect product $W_n(G) = \Sigma_n \rtimes (\pm G^n)$.

The category Monomial(G) will package together these monomial matrices $W_n(G)$ alongside projections and inclusions. An object of Monomial(G) is the $\mathbb{Z}[G]$-module $\mathbb{Z}[G][X]$ for some finite set X. A morphism in Monomial(G) is an arbitrary composition of

- inclusions, $\mathbb{Z}[G][X] \to \mathbb{Z}[X \sqcup Y]$, induced from $X \hookrightarrow X \sqcup Y$,
- projections, $\mathbb{Z}[G][X \sqcup Y] \to \mathbb{Z}[G][X]$, sending $y \in Y$ to zero, and
- monomial maps $\mathbb{Z}[G][X] \to \mathbb{Z}[G][X]$, given by an element of $W_n(G)$ when $n = |X|$.

Define a subcategory of cofibrations $\operatorname{co}\mathrm{Monomial}(G)$ by considering maps given by arbitrary compositions of inclusions and monomial maps; any such composition can be simplified to

$$Z[G][X_1] \hookrightarrow Z[G][X_1 \sqcup X_2] \cong Z[G][X_1 \sqcup X_2]$$

where the left-hand map is an inclusion induced from $X_1 \hookrightarrow X_1 \sqcup X_2$ and the right hand isomorphism is a monomial matrix in $W_n(G)$. Define a subcategory of weak equivalences $\operatorname{w}\mathrm{Monomial}(G)$ by considering only the monomial maps. Then we have

Lemma 23. *The category* Monomial(G) *with the described subcategories of cofibrations and weak equivalences is a Waldhausen category.*

Proof. (Cof 1) and (Cof 2) are clear; considering the diagram

$$\begin{array}{ccc}
\mathbb{Z}[G][X_1 \sqcup Y] & \xrightarrow{\quad f \quad} & \mathbb{Z}[G][X_1] \\
{\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\
\mathbb{Z}[G][X_1 \sqcup Y \sqcup X_2] & \longrightarrow & \mathbb{Z}[G][X_1 \sqcup X_2]
\end{array}$$

verifies the co-base change axiom (Cof 3) when the top arrow is a projection; an analogous argument verifies that (Cof 3) holds when the top arrow is an inclusion or a monomial map.

It is immediate that every isomorphism is a weak equivalence, so (Weq 1) holds. That weak equivalences can be glued follows directly by considering a few elementary cases; thus (Weq 2) holds. □

Because Monomial(G) is a Waldhausen category, we can apply the $\mathrm{S}_\bullet$ construction to produce

$$K(\mathrm{Monomial}(G)) = \Omega\left|\mathrm{wS}_\bullet \mathrm{Monomial}(G)\right|.$$

But the identification of Waldhausen's $\mathrm{S}_\bullet$ construction with Quillen's Q-construction and the Barratt–Priddy–Quillen–Segal theorem yields

$$K(\mathrm{Monomial}(G)) \simeq \Omega B\left(\coprod_{n\geq 0} BW_n(G)\right) \simeq \mathbb{Z} \times BW_\infty(G)^+ \simeq \Omega^\infty\Sigma^\infty BG_+.$$

4.2 Pairing

In Section 1.5 of [6], Waldhausen describes how to build external pairings of categories with cofibrations and weak equivalences.

Proposition 24. *Suppose the functor $F : \mathbf{A}\times\mathbf{B} \to \mathbf{C}$ is* bi-exact*, meaning that $F(A,-)$ and $F(-,B)$ are exact functors for fixed objects A of $\mathbf{A}$ and B of $\mathbf{B}$, respectively. Additionally,* *suppose that for every pair of cofibrations $A \hookrightarrow A'$ in $\operatorname{co}\mathbf{A}$ and $B \hookrightarrow B'$ in $\operatorname{co}\mathbf{B}$, the induced map*

$$F(A', B) \cup_{F(A,B)} F(A, B') \to F(B, B')$$

is a cofibration in $\mathbf{C}$. Such a functor F induces a map of bisimplicial bicategories

$$\mathrm{wS}_\bullet\mathbf{A} \times \mathrm{wS}_\bullet\mathbf{B} \to \mathrm{wwS}_\bullet\mathrm{S}_\bullet\mathbf{C}$$

and further produces a map of spaces $K(\mathbf{A}) \wedge K(\mathbf{B}) \to K(\mathbf{C})$ which extends to a map of spectra $\mathbb{K}(\mathbf{A}) \wedge \mathbb{K}(\mathbf{B}) \to \mathbb{K}(\mathbf{C})$.

We now apply Proposition 24 to produce a pairing

$$\mathbb{K}(\mathrm{Monomial}(G)) \wedge \mathcal{B}\mathbb{K}(\mathbb{Z}) \to \mathcal{B}\mathbb{K}(\mathbb{Z}[G])$$

by exhibiting a suitable biexact functor

$$F : \mathrm{Monomial}(G) \times \mathcal{B}\mathbf{Ch}_{h\mathrm{fin}}(\mathcal{B}\mathbf{P}^{\mathrm{w}}(\mathbb{Z})) \to \mathcal{B}\mathbf{Ch}_{h\mathrm{fin}}(\mathcal{B}\mathbf{P}^{\mathrm{w}}(\mathbb{Z}[G])).$$

Given $M \in \mathrm{Monomial}(G)$ and $C_\star \in \mathcal{B}\mathbf{Ch}_{h\mathrm{fin}}(\mathcal{B}\mathbf{P}^{\mathrm{w}}(\mathbb{Z}))$, define $F(M, C_\star)$ to be the chain complex $D_\star$ with

$$D_n = M \otimes_{\mathbb{Z}} C_n.$$

For a fixed $M \in \mathrm{Monomial}(G)$ or $C_\star \in \mathcal{B}\mathbf{Ch}_{h\mathrm{fin}}(\mathcal{B}\mathbf{P}^{\mathrm{w}}(\mathbb{Z}))$, the partial functors $F(M, -)$ and $F(-, C_\star)$ are exact; in other words,

- $F(-, *) = F(*, -) = *$,
- $F(M, -)$ and $F(-, C_\star)$ send weak equivalences to weak equivalences,
- $F(M, -)$ and $F(-, C_\star)$ send cofibrations to cofibrations, and
- $F(M, -)$ and $F(-, C_\star)$ preserve the pushouts appearing in (Cof 3).

There is also a technical condition to verify: given cofibrations $M \hookrightarrow M'$ in $\operatorname{co}\mathrm{Monomial}(G)$ and $C_\star \hookrightarrow C'_\star$ in $\operatorname{co}\mathcal{B}\mathbf{Ch}_{h\mathrm{fin}}(\mathcal{B}\mathbf{P}^{\mathrm{w}}(\mathbb{Z}))$, is the map

$$F(M', C_\star) \cup_{F(M, C_\star)} F(M, C'_\star) \to F(M', C'_\star)$$

a cofibration? In fact it is. The cofibrations give rise to splittings $C'_p \cong C_p \oplus C''_p$ and $M' \cong M \oplus M''$, so the degree p component of $F(M', C_\star) \cup_{F(M,C_\star)} F(M, C'_\star)$ is

$$(M' \otimes C_\star)_p \oplus_{(M \otimes C_\star)_p} (M \otimes C'_\star)_p,$$

or equivalently

$$((M \oplus M'') \otimes C_p) \oplus_{M \otimes C_p} \left(M \otimes \left(C_p \oplus C''_p\right)\right),$$

which expands to

$$(M \otimes C_p) \oplus (M'' \otimes C_p) \oplus \left(M \otimes C''_p\right),$$

and so the map into

$$F(M', C'_\star)_p = (M' \otimes C'_\star)_p = (M \oplus M'') \otimes \left(C_p \oplus C''_p\right)$$

is a cofibration, as required by the hypotheses of Proposition 24. Therefore, we have proved

Proposition 25. *The functor F induces a pairing on the level of spectra*

$$\mathbb{K}(\text{Monomial}(G)) \wedge \mathcal{B}\mathbb{K}(\mathbb{Z}) \to \mathcal{B}\mathbb{K}(\mathbb{Z}[G]),$$

which we denote by $\text{Asm}(G)$*.*

4.3 Whitehead spectrum

The assembly map

$$\text{Asm}(G) : \Sigma^\infty BG_+ \wedge \mathcal{B}\mathbb{K}(\mathbb{Z}) \to \mathcal{B}\mathbb{K}(\mathbb{Z}[G])$$

permits us to define a $\mathcal{B}$-bounded Whitehead spectrum,

$$\mathcal{B}\mathbb{W}\text{h}(G) = \text{cofiber}\,\text{Asm}(G).$$

Consider the following diagram:

$$\begin{array}{ccccc}
\Sigma^\infty BG_+ \wedge \mathcal{B}\mathbb{K}^{\text{rel}}(\mathbb{Z}) & \longrightarrow & \mathcal{B}\mathbb{K}^{\text{rel}}(\mathbb{Z}[G]) & \longrightarrow & \mathcal{B}\mathbb{W}\text{h}^{\text{rel}}(G) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^\infty BG_+ \wedge \mathcal{B}\mathbb{K}(\mathbb{Z}) & \longrightarrow & \mathcal{B}\mathbb{K}(\mathbb{Z}[G]) & \longrightarrow & \mathcal{B}\mathbb{W}\text{h}(G) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^\infty BG_+ \wedge \mathbb{K}(\mathbb{Z}) & \longrightarrow & \mathbb{K}(\mathbb{Z}[G]) & \longrightarrow & \mathbb{W}\text{h}(G)
\end{array}$$

By the functoriality of the splitting $\mathcal{B}\mathbb{K}(-) \simeq \mathbb{K}(-) \vee \mathcal{B}\mathbb{K}^{\text{rel}}(-)$, the fiber of the vertical arrow $\mathcal{B}\mathbb{W}\text{h}(G) \to \mathbb{W}\text{h}(G)$ can be identified with the cofiber $\mathcal{B}\mathbb{W}\text{h}^{\text{rel}}(G)$ of

$$\text{Asm}(G) : \Sigma^\infty BG_+ \wedge \mathcal{B}\mathbb{K}^{\text{rel}}(\mathbb{Z}) \to \mathcal{B}\mathbb{K}^{\text{rel}}(\mathbb{Z}[G])$$

and further, the following theorem holds.

Theorem 26. *There is a functorial splitting*

$$\mathcal{B}\mathbb{W}\text{h}(-) \simeq \mathbb{W}\text{h}(-) \times \mathcal{B}\mathbb{W}\text{h}^{\text{rel}}(-).$$

4.4 Concluding Remarks

The assembly map focuses attention on $\mathcal{B}K_\star(\mathbb{Z})$, which we conjecture to be highly nontrivial, even in degree zero. To illustrate some of the complexities involved, consider the map of weighted sets

$$f : (\mathbb{N}, \text{id}) \to (\mathbb{N}, \log)$$

where $(\mathbb{N}, \text{id})$ is the weighted set in which n has weight n, $(\mathbb{N}, \log)$ is the weighted set in which n has weight $\log n$, and $f(n) = n$. The map f is polynomially bounded, but the inverse is not. This map f gives rise to a map of weighted $\mathbb{Z}$-modules

$$\mathbb{Z}[\mathbb{N}] \to \mathbb{Z}[\mathbb{N}]$$

which is unboundedly an isomorphism, but not invertible as a polynomially bounded map. For the polynomial bounding class $\mathcal{P}$, the group $\mathcal{P}K_0(\mathbb{Z})$ includes classes arising from finitely generated free $\mathbb{Z}$-modules but also a class for the chain complex

$$0 \to \mathbb{Z}[\mathbb{N}] \to \mathbb{Z}[\mathbb{N}] \to 0.$$

We conjecture that the class of this chain complex is a nonzero element of infinite order in $\mathcal{P}K_0(\mathbb{Z})$.

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