

# FINITENESS PROPERTIES FOR SOME RATIONAL POINCARÉ DUALITY GROUPS

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ABSTRACT. A combination of Bestvina–Brady Morse theory and an *acyclic* reflection group trick produces a torsion-free finitely presented  $\mathbb{Q}$ -Poincaré duality group which is not the fundamental group of an aspherical closed ANR  $\mathbb{Q}$ -homology manifold.

The acyclic construction suggests asking which  $\mathbb{Q}$ -Poincaré duality groups act freely on  $\mathbb{Q}$ -acyclic spaces (i.e., which groups are  $\text{FH}(\mathbb{Q})$ ). For example, the orbifold fundamental group  $\Gamma$  of a good orbifold satisfies  $\mathbb{Q}$ -Poincaré duality, and we show  $\Gamma$  is  $\text{FH}(\mathbb{Q})$  if the Euler characteristics of certain fixed sets vanish.

## 1. INTRODUCTION

Existence and uniqueness questions (i.e., “Borel conjecture”) for closed aspherical  $\mathbb{Z}$ -homology manifolds can be formulated for  $R$ -homology manifolds. Mike Davis does this in [Dav00]: he asks if some algebra (i.e., having  $R$ -Poincaré duality) is necessarily a consequence of some geometry (i.e., being an  $R$ -homology manifold).

*Question 1.1* (M. Davis). Is every torsion-free finitely presented group satisfying  $R$ -Poincaré duality the fundamental group of an aspherical closed  $R$ -homology  $n$ -manifold?

Theorem 2.1 answers this question in the negative. However, the construction of that counterexample, as well as the spirit of the original question, suggests weakening the conclusion.

*Question 1.2* (Acyclic variant of a question of M. Davis). Suppose  $\Gamma$  is a finitely presented group satisfying  $R$ -Poincaré duality. Is there a closed  $R$ -homology manifold  $M$ , with

- $\pi_1 M = \Gamma$ , and
- $H_*(\tilde{M}; R) = H_*(\text{point}; R)$ , that is,  $R$ -acyclic universal cover?

Instead of asking for an aspherical homology manifold (as in the original question), this modified question only asks that the homology manifold have  $R$ -acyclic universal cover. Nevertheless, a group acting geometrically and

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cocompactly on an  $R$ -acyclic  $R$ -homology manifold still possesses  $R$ -Poincaré duality, so an affirmative answer to Question 1.2 provides a “geometric source” for the  $R$ -Poincaré duality of a group.

For example, if a finite group  $G$  acts on a manifold  $B\pi$ , then there is an extension

$$1 \rightarrow \pi \rightarrow \Gamma \rightarrow G \rightarrow 1, \text{ with } \Gamma = \pi_1((EG \times B\pi)/G),$$

and  $\Gamma$  satisfies  $\mathbb{Q}$ -Poincaré duality. But for  $\Gamma$  to be the fundamental group of a  $\mathbb{Q}$ -homology manifold with acyclic universal cover requires, in particular, that  $\Gamma$  be the fundamental group of a *finite complex* with  $\mathbb{Q}$ -acyclic universal cover. Thus we are led to ask

*Question 1.3.* For which groups  $\Gamma$  does there exist a finite complex  $X$  with

- $\bar{H}_*(\tilde{X}; R) = 0$  and
- $\pi_1 X = \Gamma$ ?

In other words, which groups act “nicely” (e.g., properly discontinuously, cellularly, cocompactly) on acyclic complexes? This is property  $\text{FH}(R)$  introduced by M. Bestvina and N. Brady.

Answering this question in the context of orbifolds amounts to a finiteness obstruction. W. Lück designed an equivariant finiteness theory [Lüc89] (and there are other descriptions of equivariant finiteness obstructions in the literature [tD81, Bag79]). In section 3 we will describe an equivariant finiteness theory in a setup similar to that in [DL98], with the following result:

**Main Theorem.** *Suppose a finite group  $G$  acts on a finite complex  $B\pi$ ; let  $\Gamma$  be  $\pi_1((EG \times B\pi)/G)$ , that is, the orbifold fundamental group of  $(B\pi)/G$ . If, for all nontrivial subgroups  $H \subset G$ , and every connected component  $C$  of  $(B\pi)^H$ ,*

$$\chi(C) = 0,$$

*then  $\Gamma \in \text{FH}(\mathbb{Q})$ , i.e., there exists a finite CW complex  $X$  with  $\pi_1 X = \Gamma$  and  $\tilde{X}$  rationally acyclic.*

In section 4.1, we will also see that the vanishing of certain Euler characteristics is necessary, namely  $\chi((B\pi)^H)$  for cyclic subgroups  $H$ . There are some examples in section 5.

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## 2. REFLECTION GROUP TRICK

In [Dav98], M. Davis combined Bestvina–Brady Morse theory with the reflection group trick to produce Poincaré duality groups that are not finitely

presented—and therefore, not fundamental groups of aspherical manifolds. We apply this technique to rational Poincaré duality groups and rational homology manifolds, answering Question 1.1 in the negative.

**Theorem 2.1.** *There exists a torsion-free, finitely presented  $\text{PD}(\mathbb{Q})$ -group  $\Gamma$  which is not the fundamental group of an aspherical closed ANR  $\mathbb{Q}$ -homology manifold.*

The construction of such a group  $\Gamma$  proceeds as follows:

- Let  $X$  be a simply connected finite complex which is  $\mathbb{Q}$ -acyclic but not  $\mathbb{Z}$ -acyclic (for concreteness, the CW-complex  $S^2 \cup e^3$  where the attaching map is a degree two map  $S^2 \rightarrow S^2$ ).
- Apply Bestvina–Brady Morse theory [BB97] to  $X$ ; this produces a group  $G \notin \text{FP}(\mathbb{Z})$  with  $G \in \text{FH}(\mathbb{Q})$ , so  $G$  acts freely and cocompactly on a  $\mathbb{Q}$ -acyclic space; let  $K$  be the quotient of such a free action.
- Apply a variant of M. Davis’ reflection group trick [Dav83] to a thickened version of  $K$ ; after taking a cover, this produces a torsion-free group  $\Gamma$  satisfying  $\text{PD}(\mathbb{Q})$ .
- Verify that the space  $B\Gamma$  is not homotopy equivalent to a finite complex.

A closed ANR  $\mathbb{Q}$ -homology manifold is homotopy equivalent to a finite complex [Wes77]. Since  $B\Gamma$  is not homotopy equivalent to a finite complex,  $\Gamma$  cannot be the fundamental group of an aspherical closed  $\mathbb{Q}$ -homology manifold.

**2.1. Proof of Theorem 2.1.** We exhibit a torsion-free, finitely presented  $\text{PD}(\mathbb{Q})$ -group which is not the fundamental group of an aspherical finite complex, let alone a closed ANR  $\mathbb{Q}$ -homology manifold.

**2.1.1. PL Morse theory.** We begin by producing a group which satisfies a rational finiteness property, but not an integral finiteness property. Choose a simply connected finite complex  $X$  which is  $\mathbb{Q}$ -acyclic but not  $\mathbb{Z}$ -acyclic; for the sake of concreteness,

$$X = S^2 \cup_f e^3 \text{ with } f : \partial e^3 = S^2 \rightarrow S^2 \text{ degree two.}$$

Let  $L$  be a flag triangulation of  $X$ . Then consider the union of tori

$$X = \bigcup_{\sigma \in L} \prod_{v \in \sigma} S^1.$$

In the cube complex  $X$ , it is easy to check that the link of each vertex is  $L$ , and, because  $L$  is flag, this means that  $X$  is  $\text{CAT}(0)$  (see [BH99]).

The following theorem summarizes a result of Bestvina–Brady PL Morse theory [BB97]; this is a version of Morse theory designed to analyze spaces such as the above cubical complex.

**Theorem 2.2.** *Let  $L$  be a finite flag complex. Let  $A = A_L$  the associated right angled Artin group, and  $\Gamma$  the kernel of a natural map  $A_L \rightarrow \mathbb{Z}$ .*

- *If  $L$  is  $R$ -acyclic, then  $\Gamma \in \text{FH}(R)$ .*
- *If  $L$  is simply connected, then  $\Gamma$  is finitely presented.*
- *If  $L$  is not  $R$ -acyclic, then  $\Gamma \notin \text{FP}(R)$ .*

Since  $L$  is simply connected and  $\mathbb{Q}$ -acyclic, but not  $\mathbb{Z}$ -acyclic, the corresponding  $\Gamma$  is finitely presented and  $\text{FH}(\mathbb{Q})$ , but not  $\text{FP}(\mathbb{Z})$ .

The fact that  $\Gamma \in \text{FH}(\mathbb{Q})$  means there is a finite complex  $K$  with  $\pi_1 K = \Gamma$  and  $\bar{H}_*(\tilde{K}; \mathbb{Q}) = 0$ ; it is to this complex  $K$  that we apply the reflection group trick.

2.1.2. *An acyclic reflection group trick.* Mike Davis introduced his reflection group trick in [Dav83]; his excellent book [Dav08] is a great introduction to the technique.

Since  $K$  is a finite complex, there is an embedding  $K \hookrightarrow \mathbb{R}^N$  for some  $N$ ; a *regular neighborhood* of  $K \subset \mathbb{R}^N$  is manifold  $N$  with boundary  $\partial N$ . Observe that  $N$  deformation retracts to  $K$ . An introduction to the theory of regular neighborhoods can be found in [RS72, Coh69].

The reflection group trick uses a Coxeter group to glue together copies of  $N$ , transforming the manifold with boundary  $N$  to a closed manifold  $W$ . We now describe how the copies of  $N$  are glued together. Choose a flag triangulation  $L$  of  $\partial N$ ; let  $G$  be the right-angled Coxeter group associated to this flag triangulation. For each vertex  $v \in L$ , let  $D_v$  be the star of  $v$  in the barycentric subdivision  $L'$  of  $L$ . Copies of  $N$  will be glued along the “mirrors”  $D_v$ ; specifically, define

$$\tilde{W} = (N \times G) / \sim$$

where  $(x, g) \sim (x, gh)$  whenever  $x \in D_h$ . Choose a finite index torsion-free subgroup  $G'$  of  $G$ , and let  $W = \tilde{W}/G'$ . An application of Mayer-Vietoris proves

**Proposition 2.3.**  *$W$  is a closed manifold, with  $\mathbb{Q}$ -acyclic universal cover.*

Additionally,  $\pi_1 W = G' \rtimes \Gamma$  is torsion-free, since  $G'$  and  $\Gamma$  are both torsion-free (the former by assumption, the latter because it is a subgroup of an Artin group).

**Proposition 2.4.** *The space  $B\pi_1 W$  does not have the homotopy type of a finite complex.*

*Proof.* The group  $\pi_1 W = G' \rtimes \Gamma$  retracts onto  $\Gamma$ , and so,  $B\pi_1 W$  retracts onto  $B\Gamma$ . If  $B\pi_1 W$  had the homotopy type of a finite complex, then  $B\Gamma$  would be a finitely dominated complex, and so  $\Gamma \in \text{FP}(\mathbb{Z})$ . But  $\Gamma$  was constructed above (using Bestvina–Brady Morse theory) so that  $\Gamma \notin \text{FP}(\mathbb{Z})$ .  $\square$

The existence of such a group  $W$  proves Theorem 2.1, answering Question 1.1 in the negative.

## 3. MODULES OVER CATEGORIES

We work in the framework used by Davis–Lück’s of “spaces over a category” [DL98], using the equivariant algebraic  $K$ -theory developed by Lück [Lüc89]. After setting up this framework, we will describe an instant finiteness obstruction in section 3.6; this obstruction lies in  $K_0(\mathbf{Groupoids} \downarrow R\text{-Mod})$ , in contrast to the “usual” obstruction which lies in  $K_0(R\Gamma)$ . The advantage to considering the refined obstruction in  $K_0(\mathbf{Groupoids} \downarrow R\text{-Mod})$  is that it can be computed in terms of Euler characteristics of components of the fixed sets, as we’ll see in section 4.2.

## 3.1. Categories over categories.

**Definition 3.1.** Let  $\mathcal{C}$  be a small category, and  $\mathbf{Cat}$  any category; we define a category of diagrams  $\mathcal{C}\text{-Cat}$  as follows:

- $\text{Obj } \mathcal{C}\text{-Cat}$  consists of functors  $F : \mathcal{C} \rightarrow \mathbf{Cat}$ , and
- given two such functors  $F$  and  $G$ , the morphisms  $\text{Hom}_{\mathcal{C}\text{-Cat}}(F, G)$  are the natural transformations from  $F$  to  $G$ .

This is the *functor category* and is usually denoted  $\mathbf{Cat}^{\mathcal{C}}$ ; we use the alternate notation  $\mathcal{C}\text{-Cat}$ , evoking the equivariant notation as in “ $G$ -spaces.”

*Example 3.2.* The category  $\mathbf{Spaces}$  is the category of compactly generated topological spaces; an object in  $\mathcal{C}\text{-Spaces}$  is called a (covariant)  $\mathcal{C}$ -space; a contravariant  $\mathcal{C}$ -space is a covariant  $\mathcal{C}^{\text{op}}$ -space. We likewise have  $\mathcal{C}\text{-AbGroups}$  and  $\mathcal{C}\text{-R-Mod}$ , which form abelian categories and for which W. Lück has developed homological algebra [Lüc89].

Any functor  $F : \mathbf{A} \rightarrow \mathbf{B}$  induces

$$\mathcal{C}\text{-}F : \mathcal{C}\text{-}\mathbf{A} \rightarrow \mathcal{C}\text{-}\mathbf{B}$$

by sending  $A : \mathcal{C} \rightarrow \mathbf{A}$  to  $\mathcal{C}\text{-}F(A) = F \circ A$ . For instance, the fundamental groupoid functor  $\Pi : \mathbf{Spaces} \rightarrow \mathbf{Groupoids}$  induces

$$\mathcal{C}\text{-}\Pi : \mathcal{C}\text{-}\mathbf{Spaces} \rightarrow \mathcal{C}\text{-}\mathbf{Groupoids}.$$

At times, however, we would like to talk about the category of  $\mathcal{C}$ -spaces, for a varying small category  $\mathcal{C}$ . This desire is behind the following definition.

**Definition 3.3.** Let  $\mathbf{A}$  and  $\mathbf{B}$  be categories, with  $\mathbf{A}$  a subcategory of  $\mathbf{Cat}$ , the category of small categories. Then the category

$$\mathbf{A} \downarrow \mathbf{B}$$

has as objects the functors  $F : A \rightarrow \mathbf{B}$ , for  $A$  an object of  $\mathbf{A}$ ; in other words, an object of the category  $\mathbf{A} \downarrow \mathbf{B}$  consists of a choice of an object  $A \in \mathbf{A}$ , and a functor from  $A$  to  $\mathbf{B}$ .

A morphism in  $\text{Hom}_{\mathbf{A} \downarrow \mathbf{B}}(F : A \rightarrow \mathbf{B}, F' : A' \rightarrow \mathbf{B})$  consists of a functor  $H : A \rightarrow A'$  with a natural transformation from  $F$  to  $F' \circ H$ .

Although it will not be important in the sequel, note that  $\mathbf{A}$  is a 2-category (meaning a category enriched over  $\mathbf{Cat}$ ), and  $\mathbf{A} \downarrow \mathbf{B}$  is likewise a 2-category.

**3.2. Balanced products.** A construction well-known to category theorists—that of a *coend*—gives a natural transformation from a bifunctor  $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Cat}$  to a constant functor [Mac71]. We apply this in the case of  $\mathbf{Cat}$ , a monoidal category, to combine a contravariant and covariant  $\mathcal{C}$ -object over  $\mathbf{Cat}$  into an object of  $\mathbf{Cat}$ .

**Definition 3.4.** Let  $\mathbf{Cat}$  be a monoidal category with product  $\times$ ; let  $A$  and  $B$  be contravariant and covariant  $\mathcal{C}$ -objects, respectively. Then the *balanced product* of  $A$  and  $B$ , written  $A \times_{\mathcal{C}} B$ , is

$$\bigsqcup_{c \in \text{Obj } \mathcal{C}} A(c) \times B(c) / \sim$$

where  $(xf, y) \sim (x, fy)$  for  $x \in A(d)$ ,  $y \in B(c)$ , and  $f \in \text{Hom}(c, d)$ .

We will be using balanced products in the context of spaces (under cartesian product of spaces) and modules (under tensor product of modules). At first, balanced products may seem too abstract, but balanced products (and coends more generally) are abstractions of a better-known construction: geometric realization.

*Example 3.5.* Define  $\Delta$ , the simplicial category (see [May67]), where

- $\text{Obj } \Delta$  consists of totally ordered finite sets, and
- $\text{Hom}_{\Delta}(A, B)$  consists of order-preserving functions from  $A$  to  $B$ .

Further define  $\Delta$  to be the  $\Delta$ -space, sending a totally ordered finite  $A$  to

$$\Delta(A) = (|A| - 1)\text{-simplex,}$$

and an order-preserving function to the inclusion of simplices.

A *simplicial space* is a functor  $X : \Delta^{\text{op}} \rightarrow \mathbf{Spaces}$ , i.e., an object of  $\Delta^{\text{op}}\text{-}\mathbf{Spaces}$ . The balanced product of a simplicial space  $X$  with  $\Delta$  (written  $X \times_{\Delta} \Delta$ ), is the *geometric realization* of the simplicial space  $X$ .

### 3.3. Orbit category.

**Definition 3.6.** Define the *orbit category* of a group  $G$ , written  $\text{Or}(G)$ , as follows:

- $\text{Obj } \text{Or}(G) = \{G/H : H \text{ a subgroup of } G\}$ ,
- $\text{Hom}_{\text{Or}(G)}(G/H, G/K)$  is the set of  $G$ -maps between the  $G$ -sets  $G/H$  and  $G/K$ .

Naturally associated to a  $G$ -space, there are both contravariant and covariant  $\text{Or}(G)$ -spaces.

*Example 3.7.* Let  $X$  be a (left)  $G$ -space; there is a contravariant  $\text{Or}(G)$ -space

$$G/H \mapsto X^H,$$

with  $G/H \rightarrow G/H'$  sent to  $X^{H'} \subset X^H$ . Associated to  $X$ , there is also a covariant  $\text{Or}(G)$ -space

$$G/H \mapsto X/H,$$

with  $G/H \rightarrow G/H'$  sent to  $X/H \rightarrow X/H'$ .

In fact, the reverse is possible: given a contravariant  $\text{Or}(G)$ -space, we can recover a  $G$ -space.

**Proposition 3.8.** *A contravariant  $\text{Or}(G)$ -space is (naturally) a left  $G$ -space.*

*Proof.* The construction is formally similar to geometric realization (see Example 3.5).

Suppose  $X$  is a contravariant  $\text{Or}(G)$ -space. Let  $\nabla$  be the covariant  $\text{Or}(G)$ -space given by sending  $G/H$  to itself, that is, to the finite set with the discrete topology. Then

$$X \times_{\text{Or}(G)} \nabla$$

is a (left)  $G$ -space. Specifically,  $g \in G$  acts on  $X \times_{\text{Or}(G)} \nabla$  by the map  $\text{id} \times_{\text{Or}(G)} L_g$  where  $L_g : G/H \rightarrow G/H$  is left multiplication by  $g$ .  $\square$

**3.4.  $K$ -theory.** An object in  $\mathbf{Groupoids} \downarrow R\text{-Mod}$  is an “ $R[G]$ -module” for some groupoid  $G$ ; we define certain (full) subcategories of  $\mathbf{Groupoids} \downarrow R\text{-Mod}$ , corresponding to finitely generated free and finitely generated projective  $R[G]$ -modules. We will speak of both contravariant and covariant  $R[G]$ -modules.

**Definition 3.9.** A *complex* in  $\mathbf{Groupoids} \downarrow R\text{-Mod}$  is a collection of such modules  $M_i$  (with  $i \in \mathbb{Z}$ ) and maps  $d_i : M_i \rightarrow M_{i-1}$ . We say the complex is *bounded* if all but finitely many of the modules are zero.

Write  $\mathbf{Cplx}(\mathbf{Groupoids} \downarrow R\text{-Mod})$  for the category of complexes of finitely generated projective  $R$ -modules over a groupoid; maps between complexes are *chain maps*.

As is usually the case, “free” is adjoint to “forgetful” (i.e., the forgetful functor from  $\mathbf{Groupoids} \downarrow R\text{-Mod}$  to  $\mathbf{Groupoids} \downarrow \mathbf{Sets}$ ).

**Definition 3.10** (see page 167, [Lüc89]). A module  $M$  in  $\mathbf{Groupoids} \downarrow R\text{-Mod}$  is a *free module* with basis  $B \subset M$ , an object in  $\mathbf{Groupoids} \downarrow \mathbf{Sets}$ , if, for any object  $N$  in  $\mathbf{Groupoids} \downarrow R\text{-Mod}$  and map  $f : B \rightarrow N$ , there is a unique morphism  $F : M \rightarrow N$  extending  $f$ .

In addition to free modules with basis  $B$ , we can speak about modules generated by a particular subset.

**Definition 3.11** (see page 168, [Lüc89]). Suppose  $M$  is an object in the category  $\mathbf{Groupoids} \downarrow R\text{-Mod}$ , and  $S$  is a subset (i.e., an object in  $\mathbf{Groupoids} \downarrow \mathbf{Sets}$ ). Then the *span* of  $S$  is the smallest module containing  $S$ , namely,

$$\text{span } S = \bigcap \{ N : S \subset N \text{ and } N \text{ is a submodule of } M \}.$$

If  $S$  is a finite set (i.e., finite over the indexing category, meaning  $S(g)$  is a finite set for each object  $g$  in the groupoid), we say that  $\text{span } S$  is *finitely generated*.

**Definition 3.12** (see page 169, [Lüc89]). A module  $P$  in  $\mathbf{Groupoids} \downarrow R\text{-Mod}$  is *projective* if either of the following equivalent conditions holds:

- Each exact sequence  $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$  splits.
- $P$  is a direct summand of a free module.

Having studied these modules, we can define an appropriate  $K$ -theory for the category  $\mathbf{Groupoids} \downarrow R\text{-Mod}$ , via Waldhausen categories [Wal85] as in [Lüc89]. This  $K$ -theory is the correct receiver for the Euler characteristic.

**Definition 3.13.** The *Euler characteristic*  $\chi$  of a bounded complex  $(M_i, d_i)$  in  $\mathbf{Cplx}(\mathbf{Groupoids} \downarrow R\text{-Mod})$  is

$$\chi(\cdots \rightarrow M_0 \rightarrow \cdots) = \sum_{i \in \mathbb{Z}} (-1)^i [M_i] \in K_0(\mathbf{Groupoids} \downarrow R\text{-Mod}).$$

**3.5. Chain complex of the universal cover.** In Wall's finiteness obstruction for a space  $X$ , the most important object is  $\tilde{C}(X)$ , the  $R[\pi_1 X]$ -chain complex of the universal cover of  $X$ . This is traditionally denoted by  $C_*(\tilde{X}; R)$ , but we will write  $\tilde{C}(X)$  to emphasize the functorial nature of the construction.

However, the usual construction is insufficiently functorial:  $\tilde{C}$  transforms a space  $X$  into a chain complex over a ring that depends on the group  $\pi_1 X$ ; consequently, it is not clear what the target category of  $\tilde{C}$  ought to be. Worse, only basepoint preserving maps  $X \rightarrow X$  induce endomorphisms of  $\tilde{C}(X)$ .

The definition of  $\mathbf{A} \downarrow \mathbf{B}$  is exactly what we need to define the target of the functor  $\tilde{C}$ , and by using the fundamental groupoid instead of the fundamental group, we avoid the basepoint issue: *any* self-map of  $X$  will induce a self-map of  $\tilde{C}(X)$ .

Before we can define  $\tilde{C}$ , we define the *universal cover functor*. The functor  $\tilde{\phantom{C}} : \mathbf{Spaces} \rightarrow \mathbf{Groupoids} \downarrow \mathbf{Spaces}$  sends a space  $X$  to the functor  $\tilde{X} : \Pi X \rightarrow \mathbf{Spaces}$ . This latter functor sends an object in  $\Pi X$ , which is just a point  $x \in X$ , to the universal cover of  $X$  using  $x$  as the base point.

The functor  $C : \mathbf{Spaces} \rightarrow \mathbf{Cplx}(R\text{-Mod})$  sends a space to its singular  $R$ -chain complex. Note that this induces a functor

$$\mathbf{Groupoids} \downarrow C : \mathbf{Groupoids} \downarrow \mathbf{Spaces} \rightarrow \mathbf{Groupoids} \downarrow \mathbf{Cplx}(R\text{-Mod})$$

Not too surprisingly, we compose  $\tilde{\phantom{C}}$  and  $\mathbf{Groupoids} \downarrow C$ .

**Definition 3.14** (See page 259, [Lüc89]). The functor

$$\tilde{C} : \mathbf{Spaces} \rightarrow \mathbf{Cplx}(\mathbf{Groupoids} \downarrow R\text{-Mod})$$

sends a space  $X$  to  $C(\tilde{X})$ .



Note that there is a natural map

$$\mathbf{Groupoids} \downarrow \mathbf{Cplx}(R\text{-Mod}) \rightarrow \mathbf{Cplx}(\mathbf{Groupoids} \downarrow R\text{-Mod}).$$

As a result of the functoriality of  $\tilde{C}$ , we can apply  $\tilde{C}$  over a small category  $\mathcal{C}$  to get

$$\begin{aligned} \mathcal{C}\text{-}\tilde{C} : \mathcal{C}\text{-Spaces} &\rightarrow \mathcal{C}\text{-Cplx}(\mathbf{Groupoids} \downarrow R\text{-Mod}) \\ &\rightarrow \mathbf{Cplx}(\mathcal{C} \downarrow (\mathbf{Groupoids} \downarrow R\text{-Mod})). \end{aligned}$$

**3.6. Instant finiteness obstruction.** Our goal is to define maps

$$\begin{aligned} \text{Wall} : \mathbf{FindomSpaces} &\rightarrow K_0(\mathbf{Groupoids} \downarrow R\text{-Mod}), \\ \widetilde{\text{Wall}} : \mathbf{FindomSpaces} &\rightarrow \tilde{K}_0(\mathbf{Groupoids} \downarrow R\text{-Mod}), \end{aligned}$$

so that  $\widetilde{\text{Wall}} \neq 0$  obstructs an  $R$ -finitely dominated space from being  $R$ -homotopy equivalent to a finite complex. There are a few terms that need to be defined.

Here,  $\mathbf{FindomSpaces}$  is built from a full subcategory of  $\mathbf{Spaces}$ , consisting of those spaces which are *R-finitely dominated*, but the choice of domination is part of the data.

**Definition 3.15.** A space  $Y$  is *R-dominated* by  $X$  if there are maps

$$Y \xrightarrow{i} X \xrightarrow{r} Y$$

with  $r \circ i : Y \rightarrow Y$  an  $R$ -homotopy equivalence. Further, a space is *R-finitely dominated* by  $X$  if  $X$  is a finite complex.

Whenever we speak of an  $R$ -homotopy equivalence, we really mean an  $R[\pi_1]$  equivalence—i.e., the induced map  $\tilde{C}(Y) \rightarrow \tilde{C}(Y)$  is chain homotopic to the identity.

Ranicki has defined an *instant finiteness obstruction*. His algebraic framework remains applicable for complexes of modules over a category. In particular, Proposition 3.1 of [Ran85] associates an element of  $K_0(\mathbf{Groupoids} \downarrow R\text{-Mod})$  to a finite domination; this defines the maps  $\text{Wall}$  and  $\widetilde{\text{Wall}}$  for a finitely dominated space, and Proposition 3.2 of [Ran85] yields

**Proposition 3.16.** *If a space  $X$  is  $R$ -finitely dominated, and  $\widetilde{\text{Wall}}(X) = 0$ , then  $X$  is  $R$ -homotopy equivalent to a finite complex.*

Or rather, we can only show that  $\tilde{C}(X)$  is chain equivalent to a complex of finitely generated free  $R$ -modules. But in many cases, this is enough: Leary (in Theorem 9.4 of [Lea02]), shows that if  $G$  is a group of finite type (i.e.,  $BG$  has finitely many cells in each dimension), then  $G$  is  $\text{FL}(\mathbb{Q})$  if and only if  $G$  is  $\text{FH}(\mathbb{Q})$ . This means that finite domination and the above algebra suffices to get the geometry.

What we have done thus far for spaces is valid for  $\mathcal{C}$ -spaces. For instance, a  $\mathcal{C}$ -space  $Y$  is said to be *R-finitely dominated* if there is a finite  $\mathcal{C}$ -space

$X$ , (meaning for each  $c \in \mathcal{C}$ , the space  $X(c)$  is a finite complex), and maps  $Y \xrightarrow{i} X \xrightarrow{r} Y$  with an  $R$ -homotopy equivalence  $r \circ i$ .

**Proposition 3.17.** *If  $Y$  and  $Y'$  are  $R$ -finitely dominated contra- and covariant (respectively)  $\mathcal{C}$ -spaces and  $\mathcal{C}$  is finite, then*

$$Y \times_{\mathcal{C}} Y'$$

*is an  $R$ -finitely dominated space.*

*Proof.* This is fairly straightforward: suppose  $Y$  and  $Y'$  are finitely dominated by  $X$  and  $X'$ , respectively. Then we have

$$Y \times_{\mathcal{C}} Y' \xrightarrow{i \times_{\mathcal{C}} i'} X \times_{\mathcal{C}} X' \xrightarrow{r \times_{\mathcal{C}} r'} Y \times_{\mathcal{C}} Y'$$

and it is enough to prove that

$$(r \times_{\mathcal{C}} r') \circ (i \times_{\mathcal{C}} i') : \tilde{C}(Y \times_{\mathcal{C}} Y') \rightarrow \tilde{C}(Y \times_{\mathcal{C}} Y')$$

is an  $R$ -equivalence, and that  $X \times_{\mathcal{C}} X'$  is a finite complex.  $\square$

The finiteness obstruction for a balanced product  $Y \times_{\mathcal{C}} Y'$  can be computed from the finiteness obstructions of the terms  $Y$  and  $Y'$ ; this amounts to an equivariant version of the Eilenberg–Zilber theorem, as in [GG99].

**Proposition 3.18.** *For  $Y$  and  $Y'$ , finitely dominated contravariant and covariant  $\mathcal{C}$ -spaces, respectively,*

$$\text{Wall}(Y \times_{\mathcal{C}} Y') = \text{Wall}(Y) \otimes_{\mathcal{C}} \text{Wall}(Y').$$

This follows from the decomposition on page 229 of [Lüc89], relating balanced tensor product of chain complexes to the balanced product of spaces.

#### 4. PROOF OF MAIN THEOREM

**4.1. A necessary condition.** Denote the Euler characteristics over a field  $R$  by  $\chi(X; R) = \sum_i (-1)^i \dim H_i(X; R)$ . The Lefschetz Fixed Point Theorem ([Hat02], [Bro71]) will obstruct some groups from satisfying  $\text{FH}(R)$ .

**Proposition 4.1.** *Suppose  $R \supset \mathbb{Q}$  is a field, and that*

$$1 \rightarrow \pi \rightarrow \Gamma \rightarrow G \rightarrow 1,$$

*with  $B\pi$  homotopy equivalent to a finite complex, and  $G$  a finite group. If there exists a compact  $X$  having  $\pi_1 X = \Gamma$  and  $R$ -acyclic  $\tilde{X}$ , then, for all nontrivial  $g \in G$ ,*

$$\chi((B\pi)^{(g)}; R) = 0.$$

*Proof.* The map  $\tilde{X}/\pi \rightarrow B\pi$  is an  $R$ -homology equivalence, and is  $G$ -equivariant (though not necessarily an equivariant homotopy equivalence). Consequently,

$$\begin{aligned} \chi\left((B\pi)^{(g)}; R\right) &= \text{tr}\left(g_* : H_*(B\pi; R) \rightarrow H_*(B\pi; R)\right) \quad (\text{by Lefschetz}) \\ &= \text{tr}\left(g_* : H_*(\tilde{X}/\pi; R) \rightarrow H_*(\tilde{X}/\pi; R)\right) \quad (\text{by } G\text{-equivariance}) \\ &= 0 \quad (\text{by freeness of the } G\text{-action on } \tilde{X}/\pi). \end{aligned}$$

□

Proposition 4.1 is strong enough to obstruct certain groups from satisfying  $\text{FH}(\mathbb{Q})$ .

*Example 4.2.* The group  $\mathbb{Z}/n\mathbb{Z}$  acts on  $\mathbb{Z}^n$  by permuting coordinates;  $\mathbb{Z}/n\mathbb{Z}$  also acts on the kernel of the map  $\mathbb{Z}^n \rightarrow \mathbb{Z}$  given by adding coordinates. Use the action on the kernel to define  $\Gamma = \mathbb{Z}^{n-1} \rtimes \mathbb{Z}/n\mathbb{Z}$ .

The action of  $\mathbb{Z}/n\mathbb{Z}$  on  $B\mathbb{Z}^{n-1} = (S^1)^{n-1}$  fixes  $n$  isolated points. Upon applying the classifying space functor  $B$ , the kernel  $\mathbb{Z}^{n-1}$  of the map  $\mathbb{Z}^n \rightarrow \mathbb{Z}$  is

$$B\mathbb{Z}^{n-1} = \{(\alpha_1, \dots, \alpha_n) \in (S^1)^n \mid \sum \alpha_i = 0\},$$

and  $\mathbb{Z}/n\mathbb{Z}$  acts on  $B\mathbb{Z}^{n-1}$  by cycling coordinates. So a fixed point of the  $\mathbb{Z}/n\mathbb{Z}$  is a point  $(\alpha, \dots, \alpha)$  with  $n\alpha = 0 \in S^1$ . There are  $n$  such solutions, namely  $\alpha = 2\pi k/n$  for  $k = 0, \dots, n-1$ . Consequently,

$$\chi\left(\left((S^1)^{n-1}\right)^{\mathbb{Z}/n\mathbb{Z}}\right) = n,$$

and hence Proposition 4.1 implies that  $\Gamma$  does not act freely on any  $\mathbb{Q}$ -acyclic complex.

**4.2. A sufficient condition.** The necessary condition given in Proposition 4.1 is not sufficient. As we will see later, a sufficient condition requires examining the Euler characteristic of connected components of other subgroups, not just the cyclic subgroups. Here, we use the machinery from Section 3 to prove the main theorem.

Let  $X$  be a  $G$ -space; we consider  $X$  to be a contravariant  $\text{Or}(G)$ -space by Proposition 3.8.

**Definition 4.3.**  $B\text{Or}(G)$  is the covariant  $\text{Or}(G)$ -space given by

$$G/H \mapsto BH = K(H, 1),$$

and sending the map  $G/H \rightarrow G/H'$  to the map  $BH \rightarrow BH'$  induced from  $H \subset H'$ .

**Proposition 4.4.** For a  $G$ -space  $X$ ,

$$X \times_{\text{Or}(G)} B\text{Or}(G)$$

is associated to the  $G$ -space  $X \times_G EG = (X \times EG)/G$ .

In light of Proposition 4.4, it may appear that the machinery of balanced products did little but complicate the usual Borel construction; as we will see in a moment, the machinery of balanced products will facilitate the calculation of an instant finiteness obstruction.

**Main Theorem.** *The finiteness obstruction  $\text{Wall}(X \times_{\text{Or}(G)} B\text{Or}(G))$  vanishes provided*

$$\chi(\text{connected component of } X^H) = 0$$

for all nontrivial subgroups  $H \subset G$ .

*Proof.* The covariant  $\text{Or}(G)$ -space which  $\mathbb{Q}$ -finitely dominates  $B\text{Or}(G)$  is a point over each orbit, and the contravariant  $\text{Or}(G)$ -space which  $\mathbb{Q}$ -finitely dominates  $X$  is simply  $X^H$  over each orbit. Thus the balanced product is finitely dominated, and it remains to calculate the finiteness obstruction

$$\text{Wall}(X \times_{\text{Or}(G)} B\text{Or}(G)) = \text{Wall}(X) \otimes_{\text{Or}(G)} \text{Wall}(B\text{Or}(G)).$$

Since each  $H$  is finite, the rational chain complex of  $BH$  can be taken to be the single module  $[\mathbb{Q}]$  in degree 0, since  $[\mathbb{Q}]$  is projective as a  $\mathbb{Q}H$  module. In short,  $\text{Wall}(B\text{Or}(G))(G/H)$  is  $[\mathbb{Q}]$ .

Since  $X$  is already finite,  $\text{Wall}(X)$  is the equivariant Euler characteristic; the equivariant Euler character as defined on page 99 of [Lüc89] is a chain complex over the component category  $\Pi_0(G, X)$ , which involves a contribution from each connected component of the fixed sets  $X^H$ . Provided that the Euler characteristics of the components of fixed sets of  $X$  vanish, then the balanced tensor product also vanishes, and so too the finiteness obstruction.  $\square$

## 5. EXAMPLES

There are examples where the finiteness obstruction vanishes; we give two sources of such examples: the reflection group trick, and lattices with torsion.

**5.1. Reflection group trick.** We have already seen the reflection group trick ([Dav83, Dav08]) in section 2. To further illustrate the trick, we construct a group  $\Gamma$ , with  $n$ -torsion, which is the fundamental group of a rational homology manifold (in fact, a manifold) with  $\mathbb{Q}$ -acyclic universal cover. Since the fundamental group  $\Gamma$  has  $n$ -torsion, there is no closed, aspherical manifold with fundamental group  $\Gamma$ .

**Proposition 5.1.** *Let  $G = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ . Then there exists a closed manifold  $M$  so that*

- $\pi_1 M$  retracts onto  $G$ ,
- $M$  is  $\mathbb{Q}$ -acyclic.

*Proof.* Let  $\pi = \mathbb{Z}$ , so that  $B\mathbb{Z} = S^1$ . Consider  $B\mathbb{Z}$  with the trivial  $\mathbb{Z}/n\mathbb{Z}$  action, so that the fixed set  $(B\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}} = B\mathbb{Z} = S^1$  has vanishing Euler characteristic. By the Main Theorem, there is a finite complex  $Y$  having  $\pi_1 Y = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$  and

having universal cover  $\tilde{Y}$  rationally acyclic. In other words,  $G = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \in \text{FH}(\mathbb{Q})$ . For convenience, name the generators of  $G$  by setting  $G = \langle t, s \rangle s^n = 1, st = ts$ .

For this example we do not have to apply the Main Theorem to show  $G \in \text{FH}(\mathbb{Q})$ . A construction of  $Y$ , side-stepping the Main Theorem, begins by showing that  $G \in \text{FP}(\mathbb{Q})$ , then that  $G \in \text{FL}(\mathbb{Q})$ , and finally that  $G \in \text{FH}(\mathbb{Q})$ , which yields the desired space.

First, note that  $\mathbb{Q}[\mathbb{Z}]$  is a projective  $\mathbb{Q}[G]$ -module, so

$$(1) \quad \mathbb{Q} \leftarrow \mathbb{Q}[\mathbb{Z}] \leftarrow \mathbb{Q}[\mathbb{Z}] \leftarrow 0$$

is a finite length projective resolution of  $\mathbb{Q}$  as a trivial  $\mathbb{Q}[G]$ -module; in other words,  $G \in \text{FP}(\mathbb{Q})$ . To see that  $\mathbb{Q}[\mathbb{Z}]$  is a projective  $\mathbb{Q}[G]$ -module, use the projection

$$\text{proj}_1(z) = \frac{1}{n} (1 + s + \dots + s^{n-1}) z$$

which has image isomorphic to  $\mathbb{Q}[\mathbb{Z}]$  on which  $s$  acts trivially. Let  $\mathbb{Q}[\mathbb{Z}]'$  be the image of the complementary projection,

$$\text{proj}_0(z) = \frac{1}{n} ((n-1) - s - \dots - s^{n-1}) z,$$

so that  $\mathbb{Q}[G] = \mathbb{Q}[\mathbb{Z}] \oplus \mathbb{Q}[\mathbb{Z}]'$ .

This projective resolution can be improved to a free resolution of  $\mathbb{Q}$  as a trivial  $\mathbb{Q}[G]$ -module. Tensor the resolution (1) with the resolution

$$\mathbb{Q} \leftarrow \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}] \leftarrow \mathbb{Q}' \leftarrow 0,$$

where  $\mathbb{Q}'$  is the  $\mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$ -module so that  $\mathbb{Q} \oplus \mathbb{Q}' \cong \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$ . This yields a complex

$$\mathbb{Q} \leftarrow \mathbb{Q}[G] \leftarrow \mathbb{Q}[G] \oplus \mathbb{Q}[\mathbb{Z}] \leftarrow \mathbb{Q}[\mathbb{Z}] \leftarrow 0,$$

By including a canceling pair of the complements  $\mathbb{Q}[\mathbb{Z}]'$ , we arrive at the desired free resolution

$$\mathbb{Q} \xleftarrow{\epsilon} \mathbb{Q}[G] \xleftarrow{d_1} \mathbb{Q}[G]^2 \xleftarrow{d_2} \mathbb{Q}[G] \leftarrow 0.$$

Here  $\epsilon$  is the augmentation,

$$\begin{aligned} d_1(x, y) &= (1-t)x + \text{proj}_1 y, \text{ and} \\ d_2(z) &= (\text{proj}_0 z, -(1-t)\text{proj}_1 z + \text{proj}_0 z). \end{aligned}$$

Thus,  $G \in \text{FL}(\mathbb{Q})$ .

Theorem 9.4 of [Lea02] states that a group of finite type is  $\text{FL}(\mathbb{Q})$  iff it is  $\text{FH}(\mathbb{Q})$ , so the free resolution in fact yields a space  $Y$  having  $\pi_1 Y = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$  and rationally acyclic universal cover  $\tilde{Y}$ .

Since  $Y$  is finite, it embeds in  $\mathbb{R}^N$  for some  $N$ , and we can apply the reflection group trick to a regular neighborhood of  $Y \subset \mathbb{R}^N$ , producing a manifold (not merely a rational homology manifold)  $M$  with universal cover  $\tilde{M}$  rationally acyclic, and  $\pi_1 M$  retracting onto  $G$ .  $\square$

**5.2. Preliminaries on Lattices.** Historically, the first source of Poincaré duality groups were fundamental groups of aspherical manifolds, and a basic source of aspherical manifolds are lattices.

**Proposition 5.2** ([Hel62]). *Let  $G$  be a semisimple Lie group,  $K$  a maximal compact, and  $\Gamma$  a torsion-free uniform lattice (i.e., a discrete cocompact subgroup). Then  $\Gamma \backslash G/K$  is a compact, aspherical manifold with fundamental group  $\Gamma$ .*

Consequently, uniform torsion-free lattices  $\Gamma$  satisfy  $\text{PD}(\mathbb{Z})$ . Next, we examine what happens when  $\Gamma$  is only virtually  $\text{PD}(\mathbb{Z})$ .

**Proposition 5.3.** *Let  $G$  be a finite group,  $\pi$  a group satisfying  $\text{PD}(\mathbb{Z})$ , and  $\Gamma$  an extension,*

$$1 \rightarrow \pi \rightarrow \Gamma \rightarrow G \rightarrow 1.$$

*Then  $\Gamma$  satisfies  $\text{PD}(\mathbb{Q})$ .*

One can do better than  $\mathbb{Q}$ : if  $R = \mathbb{Z}[1/G]$ , meaning  $\mathbb{Z}$  with divisors of  $|G|$  inverted, then  $\Gamma$  is  $\text{PD}(R)$ .

*Proof.* Extensions of Poincaré duality groups by Poincaré duality groups satisfy Poincaré duality [JW72], and finite groups are 0-dimensional  $\mathbb{Q}$ -Poincaré duality groups.  $\square$

Understanding groups satisfying which are virtually  $\text{PD}(\mathbb{Z})$  permits us to examine linear groups with torsion.

**Selberg's Lemma** ([Sel60]). *Every finitely generated linear group contains a finite index normal torsion-free subgroup (in other words, every such group is virtually torsion-free).*

*Example 5.4.* Uniform lattices, even when they contain torsion, satisfy  $\text{PD}(\mathbb{Q})$  because, by Selberg's lemma, a uniform lattice is virtually torsion-free, and therefore, satisfies  $\text{VPD}(\mathbb{Z})$ .

Whether  $\chi(\Gamma \backslash G/K)$  vanishes is independent of  $\Gamma$ ; it depends only on the Lie group  $G$ . This is true even if  $\Gamma$  is non-uniform (via measure equivalence [Gab02] and the equality of the  $L^2$  and usual Euler characteristic [Ati76, Lüc02]). The fixed sets are themselves lattices in smaller Lie groups, so it is easy to check that the Euler characteristic vanishes on fixed sets. As a result, lattices form a particularly nice class with respect to the finiteness obstructions in the Main Theorem. In the next section, we produce some examples.

### 5.3. Vanishing Euler characteristics of fixed sets.

**Proposition 5.5.** *For odd  $n$ , there is a uniform torsion-free arithmetic lattice  $\pi$  in  $\mathrm{SO}(n, 1)$  and a  $\mathbb{Z}/n\mathbb{Z}$  action on the locally symmetric space*

$$X = \pi \backslash \mathrm{SO}(n, 1) / \mathrm{SO}(n)$$

with fixed set  $X^{\mathbb{Z}/n\mathbb{Z}} = S^1$ .

*Proof.* We first recall the usual construction of arithmetic lattices; we follow Chapter 15C of [Mor01] and describe how to produce an arithmetic lattice in  $\mathrm{SO}(n, 1)$ . Begin by defining a bilinear form

$$B(x, y) = \sum_{i=1}^n x_i y_i - \sqrt{2} x_0 y_0$$

so that  $G = \mathrm{SO}(B) = \mathrm{SO}(n, 1)$ . Note that  $\mathbb{Z}/n\mathbb{Z}$  acts on  $\mathbb{R}^{n+1}$  preserving this form, and that the action is by integer matrices. Let  $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ , and let  $G_{\mathcal{O}}$  denote the  $\mathcal{O}$ -points of  $G$ .

The diagonal map  $\Delta : G \rightarrow G \times G^{\sigma}$ , for  $\sigma$  the Galois automorphism of  $\mathbb{Q}(\sqrt{2})$  over  $\mathbb{Q}$ , sends  $G_{\mathcal{O}}$  to  $\Delta(G_{\mathcal{O}})$ , a lattice in  $G \times G^{\sigma}$ . But  $G^{\sigma} = \mathrm{SO}(n+1)$  is compact, so after quotienting by the Galois automorphism,  $G_{\mathcal{O}}$  remains a lattice in  $G$ . And  $G_{\mathcal{O}}$  is cocompact, by the Godement Compactness Criterion (that arithmetic lattices are cocompact precisely when they have no nontrivial unipotents [MT62]). By Selberg’s lemma, we may choose a finite-index subgroup of  $G_{\mathcal{O}}$ ; let  $\pi$  denote the intersection of translates of this finite-index subgroup under the  $\mathbb{Z}/n\mathbb{Z}$  action.

The action of  $\mathbb{Z}/n\mathbb{Z}$  on  $\mathrm{SO}(n, 1)$  descends to the quotient

$$X = \pi \backslash \mathrm{SO}(n, 1) / \mathrm{SO}(n),$$

since it preserves the lattice  $\pi$ . In the universal cover  $\mathrm{SO}(n, 1) / \mathrm{SO}(n)$ , the set fixed by  $\mathbb{Z}/n\mathbb{Z}$  is a line; in the quotient manifold  $X$ , the set fixed by  $\mathbb{Z}/n\mathbb{Z}$  is no more than 1 dimensional. It is possible that the action might also have some isolated fixed points—but there are no isolated fixed points, because  $\mathbb{Z}/n\mathbb{Z}$  cannot fix isolated points on an odd-dimensional manifold (lest it freely act on the link, an even-dimensional sphere). So the fixed set  $X^{\mathbb{Z}/n\mathbb{Z}}$  is a 1-manifold, i.e., a disjoint union of circles.  $\square$

By Proposition 5.5, there is an extension

$$1 \rightarrow \pi \rightarrow \Gamma \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 1$$

and since  $\chi(B\pi^{\mathbb{Z}/n\mathbb{Z}}) = \chi(S^1) = 0$ , the equivariant finiteness theory implies that there exists a space  $Y$  with  $\pi_1 Y = \Gamma$  and whose the universal cover  $\tilde{Y}$  is a rationally acyclic space. In short,  $\Gamma \in \mathrm{FH}(\mathbb{Q})$ .

*Question 5.6.* For which  $n$  does  $\mathbb{Z}/p\mathbb{Z}$  act with nontrivial fixed set on an hyperbolic  $n$ -manifold?

This is possible in dimensions 2 and 3 by taking branched covers (as in [GT87]). Asking for a nontrivial fixed set is important: Belolipetsky and Lubotzky [BL05] have shown that for  $n \geq 2$ , every finite group acts *freely* on a compact hyperbolic  $n$ -manifold.

In contrast, the construction in Proposition 5.5 required dimension at least  $n$  to get  $\mathbb{Z}/n\mathbb{Z}$  to act with nontrivial fixed set. The fact that there are only finitely many arithmetic triangle groups [Tak77] is perhaps relevant to answering Question 5.6.

If we relax Question 5.6 to a combinatorial curvature condition (i.e., locally CAT(-1); see [BH99]), we can easily prove the following.

**Proposition 5.7.** *For every  $p$  and odd  $n \geq 3$ , there is a locally CAT(-1) manifold  $M$  admitting a  $\mathbb{Z}/p\mathbb{Z}$  action having fixed set  $M^{\mathbb{Z}/p\mathbb{Z}}$  a disjoint union of circles.*

*Proof.* In brief, first construct an action of  $\mathbb{Z}/p\mathbb{Z}$  on a closed  $n$ -manifold  $X^n$ , having fixed set a disjoint union of circles, and finish by hyperbolizing. Now we spell out a few details.

Our construction of  $X^n$  depends on our assumption that  $n$  is odd; in this case,  $\mathbb{Z}/p\mathbb{Z}$  acts freely on the odd-dimensional sphere  $S^{n-2}$ , and by taking the join with a circle on which  $\mathbb{Z}/p\mathbb{Z}$  acts trivially, we get an action of  $\mathbb{Z}/p\mathbb{Z}$  on  $S^n = S^{n-2} * S^1$  having fixed set  $S^1$ .

Triangulate  $X$  equivariantly; consequently, the fixed set  $S^1$  is in the 1-skeleton of  $X$ .

Now apply strict hyperbolization [CD95, DJ91] to the triangulation of  $X$ . The hyperbolized space inherits a  $\mathbb{Z}/p\mathbb{Z}$  action (since hyperbolization is functorial with respect to injective simplicial maps). The 1-skeleton of the hyperbolization of  $X$  consists of two copies of  $X^{(1)}$ , so a fixed circle in  $X$  contributes two circles to the hyperbolization.  $\square$

We cannot do something similar for even dimensional manifolds (because a  $\mathbb{Z}/p\mathbb{Z}$  action with circle fixed set would give, by considering the link of a fixed point, a  $\mathbb{Z}/p\mathbb{Z}$  action on an odd dimensional sphere with two fixed points, which is not possible). But by crossing the output of Proposition 5.7 with  $S^1$  on which  $\mathbb{Z}/p\mathbb{Z}$  acts trivially, we produce an even dimensional CAT(0) manifold with a  $\mathbb{Z}/p\mathbb{Z}$  action fixing a disjoint union of tori. That is, we have shown

**Corollary 5.8.** *For every  $p$  and every  $n \geq 3$ , there is a locally CAT(0) manifold  $M$  admitting a  $\mathbb{Z}/p\mathbb{Z}$  action having non-empty fixed set with vanishing Euler characteristic.*

#### REFERENCES

- [Ati76] M. F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras, Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), Soc.



- Math. France, Paris, 1976, pp. 43–72. Astérisque, No. 32–33. MR MR420729 (54 #8741)
- [Bag79] Jenny A. Baglivo, An equivariant Wall obstruction theory, Trans. Amer. Math. Soc. **256** (1979), 305–324. MR MR546920 (80k:57070)
- [BB97] Mladen Bestvina and Noel Brady, Morse theory and finiteness properties of groups, Invent. Math. **129** (1997), no. 3, 445–470. MR MR1465330 (98i:20039)
- [BH99] Martin R. Bridson and André Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR MR1744486 (2000k:53038)
- [BL05] Mikhail Belolipetsky and Alexander Lubotzky, Finite groups and hyperbolic manifolds, Invent. Math. **162** (2005), no. 3, 459–472. MR MR2198218 (2006k:57091)
- [Bro71] Robert F. Brown, The Lefschetz fixed point theorem, Scott, Foresman and Co., Glenview, Ill.-London, 1971. MR MR283793 (44 #1023)
- [CD95] Ruth M. Charney and Michael W. Davis, Strict hyperbolization, Topology **34** (1995), no. 2, 329–350. MR MR1318879 (95m:57034)
- [Coh69] Marshall M. Cohen, A general theory of relative regular neighborhoods, Trans. Amer. Math. Soc. **136** (1969), 189–229. MR MR248802 (40 #2052)
- [Dav83] Michael W. Davis, Groups generated by reflections and aspherical manifolds not covered by Euclidean space, Ann. of Math. (2) **117** (1983), no. 2, 293–324. MR MR690848 (86d:57025)
- [Dav98] ———, The cohomology of a Coxeter group with group ring coefficients, Duke Math. J. **91** (1998), no. 2, 297–314. MR MR1600586 (99b:20067)
- [Dav00] ———, Poincaré duality groups, Surveys on surgery theory, Vol. 1, Ann. of Math. Stud., vol. 145, Princeton Univ. Press, Princeton, NJ, 2000, pp. 167–193. MR MR1747535 (2001b:57001)
- [Dav08] ———, The geometry and topology of Coxeter groups, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR MR2360474 (2008k:20091)
- [DJ91] Michael W. Davis and Tadeusz Januszkiewicz, Hyperbolization of polyhedra, J. Differential Geom. **34** (1991), no. 2, 347–388. MR MR1131435 (92h:57036)
- [DL98] James F. Davis and Wolfgang Lück, Spaces over a category and assembly maps in isomorphism conjectures in  $K$ - and  $L$ -theory,  $K$ -Theory **15** (1998), no. 3, 201–252. MR MR1659969 (99m:55004)
- [Gab02] Damien Gaboriau, On orbit equivalence of measure preserving actions, Rigidity in dynamics and geometry (Cambridge, 2000), Springer, Berlin, 2002, pp. 167–186. MR MR1919400 (2003c:22027)
- [GG99] Marek Golasinski and Daciberg Lima Gonçalves, Generalized Eilenberg-Zilber type theorem and its equivariant applications, Bull. Sci. Math. **123** (1999), no. 4, 285–298. MR MR1697458 (2000d:55032)
- [GT87] M. Gromov and W. Thurston, Pinching constants for hyperbolic manifolds, Invent. Math. **89** (1987), no. 1, 1–12. MR MR892185 (88e:53058)
- [Hat02] Allen Hatcher, Algebraic topology, Cambridge University Press, Cambridge, 2002. MR MR1867354 (2002k:55001)
- [Hel62] Sigurdur Helgason, Differential geometry and symmetric spaces, Pure and Applied Mathematics, Vol. XII, Academic Press, New York, 1962. MR MR145455 (26 #2986)
- [JW72] F. E. A. Johnson and C. T. C. Wall, On groups satisfying Poincaré duality, Ann. of Math. (2) **96** (1972), 592–598. MR MR311796 (47 #358)

- [Lea02] Ian J. Leary, The Euler class of a Poincaré duality group, Proc. Edinb. Math. Soc. (2) **45** (2002), no. 2, 421–448. MR MR1912650 (2003k:20087)
- [Lüc89] Wolfgang Lück, Transformation groups and algebraic  $K$ -theory, Lecture Notes in Mathematics, vol. 1408, Springer-Verlag, Berlin, 1989, Mathematica Gottingensis. MR MR1027600 (91g:57036)
- [Lüc02] ———,  $L^2$ -invariants: theory and applications to geometry and  $K$ -theory, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR MR1926649 (2003m:58033)
- [Mac71] Saunders MacLane, Categories for the working mathematician, Springer-Verlag, New York, 1971, Graduate Texts in Mathematics, Vol. 5. MR MR354798 (50 #7275)
- [May67] J. Peter May, Simplicial objects in algebraic topology, Van Nostrand Mathematical Studies, No. 11, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967. MR MR222892 (36 #5942)
- [Mor01] Dave Witte Morris, Introduction to arithmetic groups, 2001.
- [MT62] G. D. Mostow and T. Tamagawa, On the compactness of arithmetically defined homogeneous spaces, Ann. of Math. (2) **76** (1962), 446–463. MR MR141672 (25 #5069)
- [Ran85] Andrew Ranicki, The algebraic theory of finiteness obstruction, Math. Scand. **57** (1985), no. 1, 105–126. MR MR815431 (87d:18014)
- [RS72] C. P. Rourke and B. J. Sanderson, Introduction to piecewise-linear topology, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. MR MR350744 (50 #3236)
- [Sel60] Atle Selberg, On discontinuous groups in higher-dimensional symmetric spaces, Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, pp. 147–164. MR MR130324 (24 #A188)
- [Tak77] Kisao Takeuchi, Arithmetic triangle groups, J. Math. Soc. Japan **29** (1977), no. 1, 91–106. MR MR429744 (55 #2754)
- [tD81] Tammo tom Dieck, Über projektive Moduln und Endlichkeitshindernisse bei Transformationsgruppen, Manuscripta Math. **34** (1981), no. 2-3, 135–155. MR MR620445 (82k:57028)
- [Wal85] Friedhelm Waldhausen, Algebraic  $K$ -theory of spaces, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419. MR MR802796 (86m:18011)
- [Wes77] James E. West, Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk, Ann. Math. (2) **106** (1977), no. 1, 1–18. MR MR451247 (56 #9534)

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